Quantum Mechanics
Day 14

Today

I. Last Time

- We identified why the harmonic oscillator potential $V = \frac{1}{2}m\omega^2 x^2$ is so ubiquitous in Nature; it is the low energy effective potential near any local minimum of an arbitrary potential.
- We introduced raising and lowering operators
$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i\hat{p} + m\omega\hat{x}\right)$$
and established the harmonic oscillator Hamiltonian
$$\hat{H} = \hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right).$$
- We defined the commutator
$$[\hat{A},\hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A},$$
and computed
$$[\hat{x},\hat{p}] = i\hbar \qquad \text{and} \qquad [\hat{a}_-,\hat{a}_+] = 1.$$
- We proved that if ψ is an energy eigenstate of the harmonic oscillator with energy E, then $(\hat{a}_\pm\psi)$ are also energy eigenstates with eigenenergies $E \pm \hbar\omega$.

II. The Prestige...Continued

This is a remarkable tool. The $\hat{a}$'s are ladder operators with $\hat{a}_+$ the 'raising operator' and $\hat{a}_-$ the 'lowering operator'. Is the ladder infinite? No!

Soon you will prove on the homework that

$$E \geq V_{\text{min}}.$$

So, there must be a state, call it ψ_0, such that

$$\hat{a}_-\psi_0 = 0.$$

Let's try to find it,

$$\frac{1}{\sqrt{2\hbar m\omega}}\left(i\frac{\hbar}{i}\frac{d}{dx} + m\omega x\right)\psi_0 = 0 \quad \Longrightarrow \quad \hbar\frac{d\psi_0}{dx} = -m\omega x\psi_0.$$

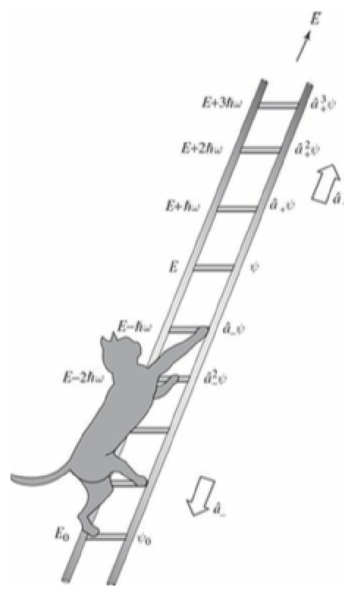

Figure 1: The ladder operators move you up and down the ladder of states for the harmonic oscillator.

This gives

$$\ln \psi_0 = -\frac{1}{2}\frac{m\omega}{\hbar}x^2 + \text{const},$$

which simplifies to

$$\psi_0 = Ae^{-\frac{m\omega x^2}{2\hbar}},$$

a very nice Gaussian! Normalizing this gives

$$\int_{-\infty}^{\infty} |A|^2 e^{-\frac{m\omega x^2}{\hbar}}\, dx = |A|^2\sqrt{\frac{\pi\hbar}{m\omega}} \quad \Longrightarrow \quad A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}.$$

Thus, the <u>ground state</u> of the harmonic oscillator is

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}.}$$

Its energy is

$$\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\psi_0 = E_0\psi_0,$$

but $\hat{a}_-\psi_0 = 0$, so

$$\frac{1}{2}\hbar\omega\psi_0 = E_0\psi_0 \quad \Longrightarrow \quad \boxed{E_0 = \frac{\hbar\omega}{2}.}$$

III. All the Oscillator States from the Ladder

Having constructed the ground state, the idea of the ladder suggests that we should be able to write

$$\boxed{\psi_n(x) = A_n(\hat{a}_+)^n\psi_0(x),}$$

for some normalization constant A_n, and

$$\boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega,}$$

gives <u>all</u> the harmonic oscillator solutions!

This works. In fact, we can even find the A_n algebraically. First

$$\hat{a}_+\psi_n = c_n\psi_{n+1} \quad \text{and} \quad \hat{a}_-\psi_n = d_n\psi_{n-1},$$

for some n-dependent constants c_n and d_n. What are these constants?

Well, note that

$$\langle f|\hat{a}_\pm g\rangle = \langle \hat{a}_\mp f|g\rangle,$$

that is, $\hat{a}_+^\dagger = \hat{a}_-$ and $\hat{a}_-^\dagger = \hat{a}_+$. <u>Proof:</u>

$$\begin{aligned}
\langle f|\hat{a}_\pm g\rangle &= \frac{1}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty} f^*\left(\mp i\hat{p} + m\omega\hat{x}\right) g\, dx \\
&= \frac{1}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{\infty}\left(\left(\mp i\hat{p} + m\omega\hat{x}\right) f\right)^* g\, dx \\
&= \langle \hat{a}_\mp f|g\rangle. \quad \blacksquare
\end{aligned}$$

Put this together with

$$\langle \hat{a}_\pm \psi_n | \hat{a}_\pm \psi_n \rangle = \langle \hat{a}_\mp \hat{a}_\pm \psi_n | \psi_n \rangle,$$

as well as

$$\hat{a}_+ \hat{a}_- \psi_n = n\psi_n, \quad \text{and} \quad \hat{a}_- \hat{a}_+ \psi_n = (n+1)\psi_n$$

to find that

$$\langle \hat{a}_+ \psi_n | \hat{a}_+ \psi_n \rangle = |c_n|^2 \langle \psi_{n+1} | \psi_{n+1} \rangle = (n+1)\langle \psi_n | \psi_n \rangle,$$

which implies

$$\boxed{c_n = \sqrt{n+1}.}$$

Similarly

$$\langle \hat{a}_- \psi_n | \hat{a}_- \psi_n \rangle = |d_n|^2 \langle \psi_{n-1} | \psi_{n-1} \rangle = n\langle \psi_n | \psi_n \rangle$$

and hence

$$\boxed{d_n = \sqrt{n}.}$$

In conclusion

$$\boxed{\begin{aligned} \hat{a}_+ \psi_n &= \sqrt{n+1}\psi_{n+1}, \\ \hat{a}_- \psi_n &= \sqrt{n}\psi_{n-1}, \end{aligned}}$$

and

$$\boxed{\psi_n = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n \psi_0.}$$

These results are so nice that we often abbreviate them even further to make the structure yet more transparent. We define

$$|n\rangle \equiv |\psi_n\rangle$$

and write

$$\boxed{\begin{aligned} \hat{a}_+ |n\rangle &= \sqrt{n+1}\,|n+1\rangle, \\ \hat{a}_- |n\rangle &= \sqrt{n}\,|n-1\rangle, \end{aligned}}$$

and

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n |0\rangle.}$$

These stationary states are orthonormal

$$\langle m|n\rangle = \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \delta_{mn}.$$

<u>Proof:</u> We compute

$$\int_{-\infty}^{\infty} \psi_m^* (\hat{a}_+ \hat{a}_-) \psi_n \, dx = n \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx,$$

but also, applying our adjoint results

$$\int_{-\infty}^{\infty} \psi_m^* (\hat{a}_+ \hat{a}_-) \psi_n \, dx = \int_{-\infty}^{\infty} (\hat{a}_- \psi_m)^* (\hat{a}_- \psi_n) \, dx$$
$$= \int_{-\infty}^{\infty} (\hat{a}_+ \hat{a}_- \psi_m)^* \psi_n \, dx = m \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx.$$

For $m \neq n$, the only way that

$$n \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = m \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx$$

is if

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = 0.$$

We defined the ψ_n such that they were normalized and hence we've proven that they are orthonormal.

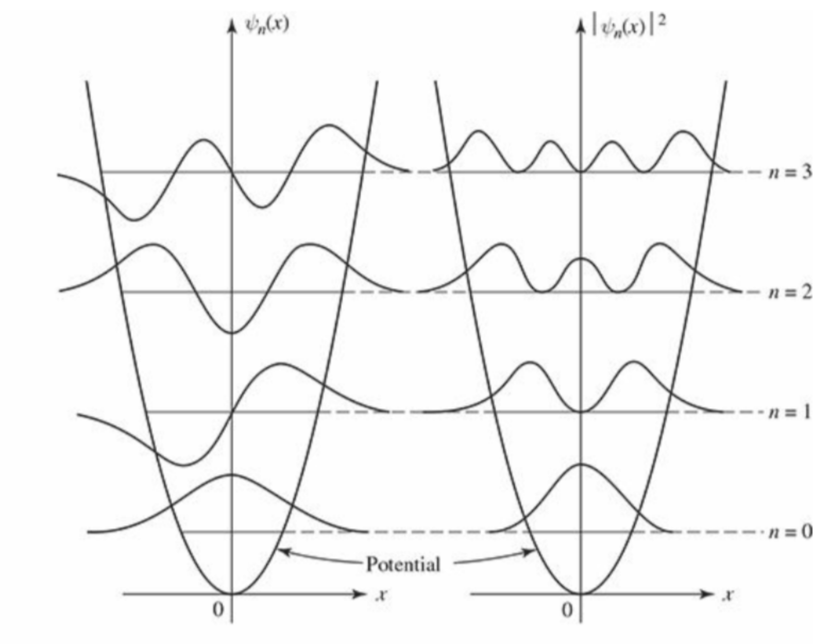

Figure 2: The first several harmonic oscillator stationary states (left) and the corresponding probability densities (right).

IV. Explicit Formulas

Much of what we've done above still remains implicit. Our one explicit result is the ground state wave function

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}, \qquad \text{and} \qquad E_0 = \frac{1}{2}\hbar\omega.$$

Let's illustrate that the above results can be made completely explicit too. For example, compute

$$\psi_1(x) = \hat{a}_+\psi_0(x) = \frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$
$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2}.$$

It's a quick check to confirm that this wave function is properly normalized.