Quantum Mechanics
Day 15

Today

I. Last Time

- We have a whole table of result for the harmonic oscillator now:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}, \quad \text{and} \quad \hat{V} = \frac{1}{2}m\omega^2\hat{x}^2,$$

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i\hat{p} + m\omega\hat{x}\right), \qquad \text{and} \qquad [\hat{a}_-, \hat{a}_+] = 1,$$

$$\hat{H} = \hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right).$$

  We also found the ground state wave function explicitly

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}.$$

- We said it in class, but perhaps it deserves a second emphasis, our computation of the canonical commutator

$$[\hat{x}, \hat{p}] = i\hbar$$

  was completely general and holds for all states on which these operators make sense.

- We introduced a shorthand for the harmonic oscillator's energy eigenstates

$$|n\rangle \equiv |\psi_n\rangle \quad \text{with} \quad \hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle,$$

  and found

$$\begin{aligned} \hat{a}_+|n\rangle &= \sqrt{n+1}\,|n+1\rangle, \\ \hat{a}_-|n\rangle &= \sqrt{n}\,|n-1\rangle, \end{aligned}$$

  and

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|0\rangle.$$

  Carefully note that for the harmonic oscillator $n = 0, 1, 2, \ldots$ instead of the $n = 1, 2, 3, \ldots$ of the infinite square well.

II. The Number Operator

To practice working with the ladder operators Zak computed

$$\hat{a}_+\hat{a}_-|n\rangle = \hat{a}_+\sqrt{n}|n-1\rangle = \sqrt{n}\hat{a}_+|n-1\rangle = \sqrt{n}\sqrt{n}|n\rangle = n|n\rangle.$$

This is such a nice result that people give this operator its own name, the number operator

$$\boxed{\hat{N} \equiv \hat{a}_+\hat{a}_-.}$$

Then Henry computed things in the other order

$$\begin{aligned}\hat{a}_-\hat{a}_+|n\rangle &= \hat{a}_-\sqrt{n+1}|n+1\rangle = \sqrt{n+1}\hat{a}_-|n+1\rangle \\ &= \sqrt{n+1}\sqrt{n+1}|n\rangle = (n+1)|n\rangle.\end{aligned}$$

This operator usually isn't named, but in class we were playfully calling it the $(n+1)$-operator. It did, however, give us another way to understand the commutation relation of the ladder operators, since

$$[\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = (\hat{N}+1) - \hat{N} = 1!$$

III. More Explicit Formulas for the Harmonic Oscillator

The algebraic tools allow us to do more than this though. Notice that so far we have thought of expressing $\hat{a}_\pm$ in terms of $\hat{x}$ and $\hat{p}$, but we can turn this logic around and write the latter operators in terms of the former. Cecily did the algebra to find

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-) \quad \text{and} \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-).$$

Using this new method, Ethan computed

$$\begin{aligned}\langle\hat{x}\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle n|(\hat{a}_+ + \hat{a}_-)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\langle n|\hat{a}_+|n\rangle + \langle n|\hat{a}_-|n\rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n+1}\langle n|n+1\rangle + \sqrt{n}\langle n|n-1\rangle) = 0,\end{aligned}$$

where the last equality follows from orthonormality of the eigenstates. Then Saiqi did the computation for $\hat{p}$,

$$\langle\hat{p}\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle n|(\hat{a}_+ - \hat{a}_-)|n\rangle = 0,$$

where this time we quickly realized that the ladder operators were going to produce orthogonal states and hence that the result vanished.

Saiqi also tackled a new sort of computation, representative of a whole class of things we can do now,

$$\langle\hat{V}\rangle = \left\langle \frac{1}{2}m\omega^2\hat{x}^2 \right\rangle = \frac{1}{2}m\omega^2\left\langle \hat{x}^2 \right\rangle.$$

Expanding the square of $\hat{x}$ gives

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left[(\hat{a}_+)^2 + (\hat{a}_+\hat{a}_-) + (\hat{a}_-\hat{a}_+) + (\hat{a}_-)^2\right].$$

Wonderfully, however,

$$\langle n|(\hat{a}_+)^2|n\rangle \propto \langle n|n+2\rangle = 0,$$

and similarly for $(\hat{a}_-)^2$. So, only the two middle terms survive and we have

$$\langle V\rangle = \frac{\hbar\omega}{4}(n+n+1) = \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right).$$

Apparently, for all the energy eigenstates of the harmonic oscillator exactly half of the total energy is in the potential and half is in the kinetic energy. This is a special result for the oscillator.