Quantum Mechanics
Day 17

Today
I. Last Time
II. The Free Particle
III. Wave Packets

I. Last Time

- As often happens with my exams, this exam was too hard. This is a fault in my process around building exams and should not be taken as your failing. Please take that statement to heart! I was pleased with what you all did on the exam and it showed lots of understanding in the direction that I hope to see you go. Keep up your hard work!
- We played the number game, see e.g. Can You Solve This? I want you to keep this game in mind as you review your exams. You learn more by getting some things right and some things wrong.
- There are at least two troubles with my exam design process: 1. I want exams to be too interesting. 2. In the pursuit of you getting some things wrong, I often make the exam too difficult. But, do keep in mind that you want to get some things wrong! If you don't, you haven't learned anything from the process.

II. The Free Particle

What does it mean for a particle to be free? $V = 0$! So,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

or

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \qquad \text{with} \quad k \equiv \frac{\sqrt{2mE}}{\hbar}.$$

Again this is the harmonic motion differential equation. But, now it's better to write

$$\psi(x) = Ae^{ikx} + Be^{-ikx}.$$

Why this difference from the square well? Here there are no boundary conditions—so E's are not quantized! We just have that the energy is directly related to the wave number, explicitly

$$E = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}.$$

The full time-dependent solution is now

$$\Psi(x,t) = Ae^{ik\left(x - \frac{\hbar k}{2m}t\right)} + Be^{-ik\left(x + \frac{\hbar k}{2m}t\right)},$$

with the first term a rightwards traveling wave and the 2nd a leftwards traveling wave. Recall, however, that any f such that $f = f(x \pm vt)$ is a wave moving in the $\mp x$ direction with speed v. (Note $x + vt = \text{const} \implies dx/dt = -v$.)

We often just write the one term

$$\Psi_k(x,t) = Ae^{ik\left(x - \frac{\hbar k}{2m}t\right)}$$

and use the two signs of the wavenumber

$$k = \pm\frac{\sqrt{2mE}}{\hbar}$$

to capture the two directions of travel for the wave. Notice that I have also added a subscript k to label the state—I am leveraging that a solution is completely characterized by its wavenumber to enrich the notation.

These stationary states have definite wavelength and momentum

$$\lambda = \frac{2\pi}{|k|} \quad \text{and} \quad p = \hbar k.$$

The speed at which they move is

$$v_{\text{quantum}} = \frac{\hbar|k|}{2m} = \sqrt{\frac{E}{2m}}.$$

Note that ($E = \frac{1}{2}mv^2$)

$$v_{\text{classical}} = \sqrt{\frac{2E}{m}} = 2v_{\text{quantum}}.$$

Apparently these stationary states are not, in themselves, modeling classical particles with a definite energy.

Even worse Ψ_k is not normalizable:

$$|A|^2\int_{-\infty}^{\infty}\Psi_k^*\Psi_k dx = |A|^2\int_{-\infty}^{\infty}dx = \infty|A|^2.$$

These solutions are still immensely useful. They span the solutions of Schrödinger's equation. We can add together solutions of different k with different multiplying weights to get any solution. There is a new twist though, since k is a continuous variable the weights actually make up a function, that is,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{ik\left(x - \frac{\hbar k}{2m}t\right)}dk.$$

III. Wave Packets

This wave packet, because it combines a range of k's, can be normalizable!

Suppose I'm given a $\Psi(x,0)$ that is normalizable, how do I find $\phi(k)$? This turns out to be a fancier version of Fourier's trick and you found the answer when you proved Plancherel's theorem:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk \iff F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

Applying Plancherel to the given $\Psi(x,0)$ we have

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx}\, dx,$$

and we can see that $\phi(k)$ determines the shape of the wave packet $\Psi(x,0)$.

Why does combining wave numbers allow wave packets to become normalizable? Let's explore this through an example:

Ex 1: Let's combine a finite number, say 3, plane waves. Let's choose k_0, $k_0 - \frac{1}{2}\Delta k$, and $k_0 + \frac{1}{2}\Delta k$ with amplitudes in the ratio $1 : 1/2 : 1/2$ then

$$\begin{aligned}\psi(x) &= \frac{A}{\sqrt{2\pi}} \left[e^{ik_0 x} + \frac{1}{2} e^{i\left(k_0 - \frac{1}{2}\Delta k\right)x} + \frac{1}{2} e^{i\left(k_0 + \frac{1}{2}\Delta k\right)x} \right] \\ &= \frac{A}{\sqrt{2\pi}} e^{ik_0 x} \left[1 + \cos\left(\frac{\Delta k}{2} x \right) \right].\end{aligned}$$

This has a maximum at $x = 0$, but decreases as x increases due to deconstructive interference between the constituent waves. The interference is completely deconstructive when the phase shift $e^{\pm i \frac{1}{2}\Delta k x}$ is -1, i.e. when

$$\pm\frac{\Delta x}{2} \left(\pm\frac{\Delta k}{2} \right) = \pi \qquad \Longrightarrow \qquad \Delta x \Delta k = 4\pi.$$

This illustrates a remarkable (and completely general) tradeoff; the more localized the wave packet (small Δx), the greater its width in wave numbers (large Δk, or Δp). Tighter bounds relating these ranges is what uncertainty principles are all about and will be our focus in the future.

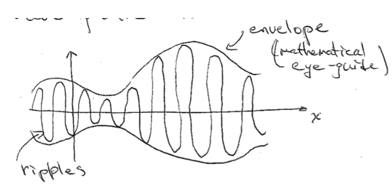

Figure 1: The structure of a wave packet. The packet is made up of the internal ripples, but interference effects cause its amplitude to vary in space and contain the wave packet. The envelope curve is not part of the wave, but just a way to guide the eye through the changes in amplitude. The envelope can be derived mathematically.

A typical structure for a wave packet is shown at right. The ripples travel at the phase velocity

$$\boxed{v_{\text{phase}} = \frac{\omega}{k}.}$$

[Aside: In the present context

$$\omega = \omega(k) = \frac{\hbar k^2}{2m},$$

but both the phase velocity above and the the argument we are about to make will work for all dispersion relations $\omega = \omega(k)$.]