Quantum Mechanics
Day 2

I. Last Time

Could Quantum Mechanics be otherwise? We restricted attention to real numbers for the moment and introduced two 'norms': the 1-norm

$$p_1 + \cdots + p_N = \sum_i p_i = 1, \qquad p_i \in [0,1],$$

and the 2-norm

$$|\alpha|^2 + |\beta|^2 + \cdots + |\omega|^2 = 1, \qquad \alpha, \beta, \cdots \in \mathbb{R}.$$

We can use the 2-norm if we interpret the *squares* as the probabilities. For example, take (α, β) as variables, we want

$$|\alpha|^2 + |\beta|^2 = 1.$$

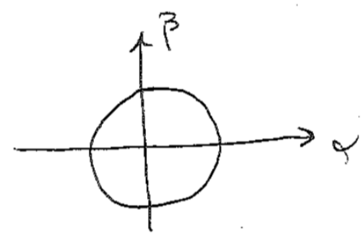

Figure 1: The plane of values of α and β. The set $|\alpha|^2 + |\beta|^2 = 1$ is the unit circle in this plane for real α and β.

All such (α, β) form a circle. But then, why not just forget about α and β and only work with $|\alpha|^2$ and $|\beta|^2$? That is, why not just return to the 1-norm?

This led us to ask "Which transformations preserve the 2-norm?" Answer: the orthogonal transformations. Matrices that represent an orthogonal transformation satisfy

$$\widetilde{\mathsf{O}}\mathsf{O} = \mathsf{I} \qquad \text{or} \qquad \widetilde{\mathsf{O}} = \mathsf{O}^{-1}.$$

We found that rotations, like

$$\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

are an example of orthogonal transformations. In fact, the full group of orthogonal transformations is made up of rotations and parity transformations, like

$$\mathsf{P} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

which is a reflection about the y-axis. (Notice that reflections also preserve the length of a vector, i.e. the 2-norm.)

II. Transformations Continued

We turn now to the 1-norm. In probability theory a valid transformation must preserve the 1-norm of your state, e.g. starting with

$$\begin{pmatrix} p \\ 1-p \end{pmatrix},$$

a bit, one valid transformation is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} 1-p \\ p \end{pmatrix},$$

which is sometimes called a bit flip. Did this result in another valid probabilistic description of the transformed bit? Yes! Because it preserved the 1-norm.

Question: What are the conditions on a completely generic matrix

$$\mathsf{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for it to give a valid transformation of a probabilistic bit? (These are called stochastic matrices.)

To investigate this question we act S on a general bit state to find

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ 1-p \end{pmatrix} = \begin{pmatrix} ap + b(1-p) \\ cp + d(1-p) \end{pmatrix}.$$

We'd like to understand what are the possible freedoms for a, b, c, and d. This transformation should work for any p, so take as an example $p = 1$. Then the final state is

$$\begin{pmatrix} a \\ c \end{pmatrix}$$

and this final state must satisfy $a \in [0, 1]$, $c \in [0, 1]$, and the 1-norm condition

$$a + c = 1 \quad \Longrightarrow \quad c = 1 - a.$$

We can also take the special case $p = 0$ to get the state

$$\begin{pmatrix} b \\ d \end{pmatrix},$$

which satisfies $b \in [0, 1]$, $d \in [0, 1]$, and the 1-norm condition if $d = 1 - b$. So, our stochastic matrix has the form

$$\mathsf{S} = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}, \qquad \text{with } a, b \in [0, 1],$$

that is, it's columns add up to 1. This generalizes to $n \times n$ stochastic matrices.

We now have a complete characterization of the transformations that preserve the 2-norm and those that preserve the 1-norm. So, can these two types of transformations lead to different physics? The answer is an emphatic yes!

III. Interference

Last time we mentioned that a quantum two-state system was called a qubit and in quantum mechanics we use the 2-norm. Now we'd like to understand if there is a physical difference between the possibilities using the 1-norm and the 2-norm.

Let's consider a quantum coin. Our matrix formalism is rich enough to encompass both outcomes of a the coin's flip. Let's interpret the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

to mean that we will definitely get heads. Similarly,

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

means we will definitely get tails. The idea of the amplitude formalism is that the general state

$$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

represents getting heads with probability $|\alpha|^2$ and tails with probability $|\beta|^2$.

Let's use our new ability to transform outcomes and rotate the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by 45°. Putting 45° into our rotation matrix gives

$$\mathsf{R}(45^\circ) = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

So our rotated state is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

The resulting amplitudes $1/\sqrt{2}$ and $1/\sqrt{2}$ correspond to a 50-50 probability for the two outcomes. We've figured out how to 'flip' a quantum coin.

Let's consider doing it again, the resulting state is

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Strikingly, now the outcome is certain. The twice transformed state definitely leads to a result of tails. By flipping a flipped coin we get a definite answer. This is a quantum interference phenomenon! Notice the essential role the minus sign played here—the two norm matters for predicting the outcome of physical experiments.