Quantum Mechanics
Day 3

Today

I. Last Time

- The heart of the difference between the 1-norm and the 2-norm is that different sets of linear *transformations* preserve these two norms.
- In particular the transformations preserving the 2-norm can have negative entries and this allows for interference phenomena!

  We didn't prove it, but m-norms with $m > 2$ are even less interesting because they are only preserved by a finite set of transformations.
- We demonstrated interference explicitly for a qubit, see Figure at right. Let's revisit this with a more physical example.

II. Physical Example

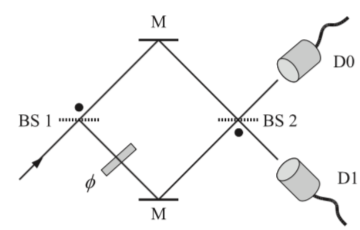

Figure 1: A Mach-Zehnder interferometer. Here BS =Beam Splitter, M =Mirror, ϕ=a wave plate, and D=a detector. The dots indicate the side of the beam splitter that is silvered.

Let's return to the Mach-Zehnder setup pictured at right.

Here the beam splitter acts as a physical transformation of the input state. If $BS1$ is a half-silvered mirror it lets through half of the light and reflects the other half.

Suppose we represent putting a photon in from above and putting one in from below by the states

$$\begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}0\\1\end{pmatrix},$$

respectively, then what is the matrix representation of the beam splitter's action?

Representing this action by a matrix we have

$$\mathsf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}.$$

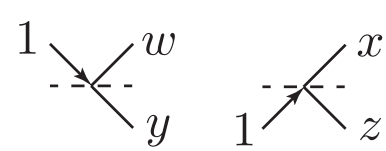

Figure 2: Action of beam splitters.

But, we already understand what a beam splitter does, it takes a definite input state and breaks it into equally probably outcomes for either output. So,

$$\mathsf{B}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}w\\y\end{pmatrix} \qquad \text{and so} \qquad |w|^2 = |y|^2 = \frac{1}{2}.$$

Similarly,

$$\mathsf{B}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}x\\z\end{pmatrix} \qquad \text{and so} \qquad |x|^2 = |z|^2 = \frac{1}{2}.$$

Notice that so far we only know the magnitudes of w, x, y, and z. However, we also know that we want the matrix to be orthogonal to really preserve the 2-norm, so

$$\mathsf{B}_\ell = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Here the subscript ℓ and the location that I have chosen for the -1 are related. With this entry negative, B_ℓ represents a beam splitter with the silvered side on the 'lower' side of the beam splitter, hence the ℓ subscript. The air-silver interface cause a π phase shift in the light and is the source of the minus sign. In the diagrams, I have indicated the silvering with a dot (•) and hence B_ℓ is the matrix representation of BS 2. The matrix for BS 1 is

Figure 3: A beam splitter with the silvering indicated explicitly by the •.

$$\mathsf{B}_u = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Question: If we now send in a photon from below

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which detector(s) see a click(s)? Quantum theory allows us to compute the answer:

$$\mathsf{B}_\ell \mathsf{B}_u \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So, only the upper detector sees a click! The same interference phenomenon we saw last time. (Notice that the order of the matrices is dictated by which beam splitter is encountered first, namely BS 1 and then BS 2. So, B_u—the representative of BS 1—acts on the input first and then B_ℓ. The algebra, as read from left to right, swaps this ordering.)

Experiments confirm this prediction of quantum mechanics!!

III. Interlude

Could Quantum Mechanics be different? We have shown that there are good reasons that quantum theory makes the move to consider amplitudes (α's and β's) instead of trying to directly predict probabilities (p's).

There are two more immediately mysterious aspects of quantum theory.

Not only do we use the 2-norm, but we allow the amplitues α, $\beta, \ldots$ to be complex numbers: $\alpha, \beta, \cdots \in \mathbb{C}$. This is why the absolute values in the 2-norm are essential

$$|\alpha|^2 + |\beta|^2 = 1.$$

Mystery #1: Why are quantum amplitudes complex?

The second mystery is an unstated assumption in all of our discussion so far. Mystery #2: Why have we restricted attention to only *linear* transformations that preserve the 2-norm?

My current favorite answers to these questions are better sutied to discussions later in the course; but, I hope you will ponder them yourselves as, particularly my answer to Mystery #1, just trades it for another mystery.

IV. Probability Theory

Soon we will transition to the wave theory of quantum mechanics. What unites both the matrix and the wave formulations is their focus on predicting probabilities. Probability theory is truly at the heart of quantum mechanics and many of the surprising features of the theory are actually just a consequence of focusing on probabilities.

Fortunately, the basics of probability theory are familiar. The foundation of the theory is just counting—no need for intimidation! On the other hand, the surprising thing about counting is that it can be hard.

Discrete systems with a finite total number of possibilities are the simplest. For them the probability of an event e is just the ratio of two counts:

$$P(\text{event } e) = \frac{\text{\# of events } e}{\text{total \# of events}},$$

here the hash symbol (#) is shorthand for 'number'.

Ex: A bag contains 3 red marbles, 2 black and 1 white. What's the probability of drawing a red marble?

$$P(\text{red}) = \frac{3 \text{ reds}}{6 \text{ total}} = \frac{1}{2}.$$

An easy (but important!) coherence check: if you draw a marble what's the probability of drawing a marble?

$$\begin{aligned} P(\text{marble}) = 1 &= P(\text{red}) + P(\text{black}) + P(\text{white}) \\ &= \frac{1}{2} + \frac{1}{3} + \frac{1}{6}. \end{aligned}$$

The first equality is boring, but the second and third are more interesting. They expose relationships between events.

⌜Aside: Here is an llustration of the challenges of probability theory: Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?

Mrs. Smith has two children. At least one of them is a boy. What is the probability that both children are boys? ⌟

In a class of 3 students a particularly difficult exam is given with the results:

Student	Score (100)
S1	42
S2	57
S3	42

What is the mean score? Using ⟨⟩ to indicate the mean and j the score, we have

$$\langle j\rangle = \frac{42+57+42}{3} = \frac{141}{3} = 47.$$

With measurements in mind, in quantum theory we call this the "expectation value." You have clear intuition for this, but let's rewrite it in a general form

$$\langle j\rangle = 57\cdot\frac{1}{3} + 42\cdot\frac{2}{3} = \sum_j j\,P(j),$$

in this way of writing it each of the fractions is the probability of that score, which is explicit in the last equality. The last formula works for *any* kind of variable

$$\langle\heartsuit\rangle = \sum_{\heartsuit's} \heartsuit\, P(\heartsuit).$$

So,

$$\langle j^2\rangle = \sum_{j's} j^2\, P(j).$$

Notice an odd property of expectation values that arises in the graph at right: we would never actually measure the expected value. This is one reason why it is often nice to characterize the spread of your data.

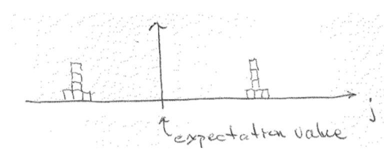

Figure 4: Some times the expectation value of a data set is a value that is not present in the data set.

A first attempt would be to define $\Delta j \equiv j - \langle j\rangle$ and compute

$$\langle\Delta j\rangle = \langle j - \langle j\rangle\rangle.$$

But,

$$\begin{aligned}\langle\Delta j\rangle &= \sum_j (j - \langle j\rangle)P(j)\\ &= \sum_j jP(j) - \sum_j \langle j\rangle P(j)\\ &= \langle j\rangle - \langle j\rangle \sum_j P(j) = 0!\end{aligned}$$

So, this won't work. Next time we will try $\langle(\Delta j)^2\rangle$, which works well!