Quantum Mechanics
Day 4

Today

I. Last Time
II. A bit more probability
III. What's to come?
IV. Fermat's Principle

I. Last Time

- Explored a physical example with quantum interference—the Mach-Zehnder interferometer.
- The 2-norm is what allows amplitudes to take center stage and, hence, makes interference possible. We left open the mysteries of complex numbers and linearity.
- We began probability theory. We defined

$$P(\text{event } e) = \frac{\#\text{ of events } e}{\text{total }\#\text{ of events}},$$

which is subject to

$$\sum_i P_i = 1.$$

We also found

$$\langle j\rangle = \sum_{j's} jP(j),$$

and its generalizations and checked that

$$\langle \Delta j\rangle \equiv \langle (j - \langle j\rangle)\rangle = 0.$$

II. A bit more probability

This brings us back to a computation that we were in the midst of doing. We wanted to find

$$\begin{aligned}\sigma^2 \equiv \langle \Delta j^2\rangle &= \sum_j (j-\langle j\rangle)^2 P(j)\\ &= \sum_j (j^2 - 2j\langle j\rangle + \langle j\rangle^2)P(j)\\ &= \sum_j j^2 P(j) - 2\langle j\rangle \sum_j jP(j) + \langle j\rangle^2 \sum_j P(j)\\ &= \langle j^2\rangle - 2\langle j\rangle\langle j\rangle + \langle j\rangle^2\\ &= \langle j^2\rangle - \langle j\rangle^2.\end{aligned}$$

The last line supplies by far the easiest way to compute the standard deviation

$$\boxed{\sigma = \sqrt{\langle j^2\rangle - \langle j\rangle^2}.}$$

This is a formula worth memorizing.

That's it, for (most) practical calculations we're only going to need $\langle j \rangle$, σ, and $P(j)$. However, we do need one more technique; that is, the ability to deal with continuous probability distributions.

Suppose you want to know how probable your height is, how would you go about finding out? You would grab a tape measure and find your height, but only with limited precision. Let's say it is 5'8"±1/16", then how probable is this?

To answer that question we need a function that tells us the likelihood of different heights and we need to integrate it. This is an exceedingly important process.

$$P(\text{your height}) = \int_{5'8''-1/16''}^{5'8''+1/16''} \rho(h) dh.$$

Notice that the increment dh is a small difference in heights, and as such, carries units: $[dh]$ =length (here feet and inches). But, if it carries units, what does that mean about $\rho(h)$? It also must carry units

$$[\rho(h)] = \frac{\text{probability}}{\text{length}}.$$

It is the denominator in this expression that leads us to call ρ and all of its cousins "probability densities".

I like the way that Griffiths (and Schroeter) write this

$$\rho(h) dh = \begin{cases} \text{Probability that an individual (chosen} \\ \text{at random) lies between } h \text{ and } h + dh. \end{cases}$$

Once you understand this, everything else is quite similar:

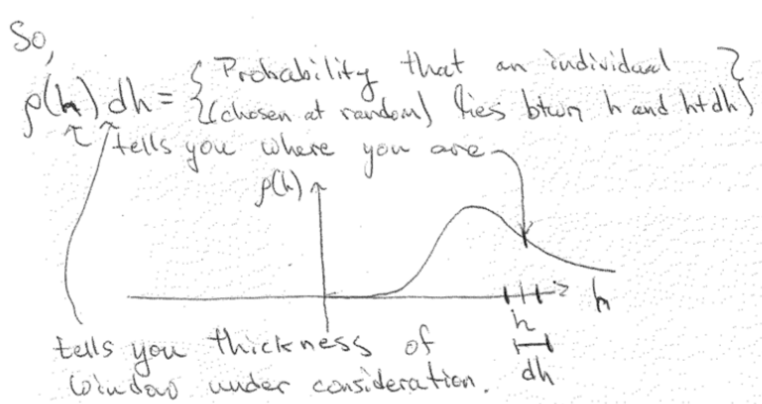

Figure 1: The meaning of the probability density.

$$\int_{-\infty}^{\infty} \rho(x)dx = 1$$
$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)dx$$
$$\langle \heartsuit(x) \rangle = \int_{-\infty}^{\infty} \heartsuit(x)\rho(x)dx$$
$$\sigma^2 \equiv \langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2.$$

As you probably remember, the wave function $\psi(x)$ is the analog of the amplitudes α, β, etc, that we have been considering; that is,

$$\boxed{\rho(x) = |\psi(x)|^2 = \psi(x)^*\psi(x).}$$

III. What's to come?

My plan is to give you a brief introduction to where the wave formulation of quantum mechanics comes from in the next 3 lectures. This is not for culture or because it's "good for you" to know the history. Instead, I want to try and motivate that many unintuitive ideas in quantum didn't come out of nowhere, but were motivated by classical mechanics. I want you to have some physical insight into the definitions we will constantly use.

IV. Fermat's principle

Light moving in a vacuum travels at a constant speed c. In a medium with index of refraction n the speed is

$$v = \frac{c}{n}.$$

In a vacuum if we send light from a source point S to a detector at point D, it travels in a straight line: What if the detector is

Figure 2: Straight-line motion of light in vacuum.

embedded in a medium, say glass, then what is the path of the ray?

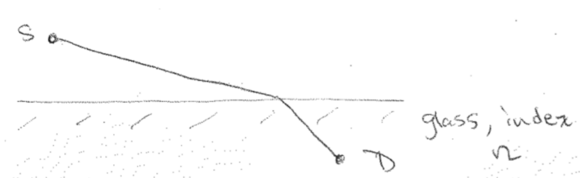

Figure 3: Light traveling from vacuum into glass.

The answer is familiar from Snell's law, but Fermat gave a very interesting formulation

> **Fermat's principle**: Light travels the path from S to D that is extremal in time.

To test what Fermat's principle says about this situation, let's put coordinates on our picture From the figure we can compute the

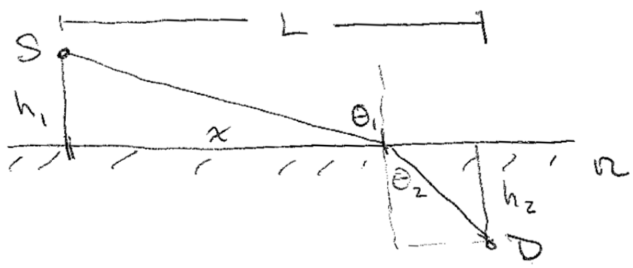

Figure 4: Light departs from a source in vacuum at S and arrives at a detector D in glass. What path does it take?

travel times in each medium

$$t_1 = \frac{\sqrt{h_1^2 + x^2}}{c} \qquad \text{and} \qquad t_2 = \frac{\sqrt{h_2^2 + (L-x)^2}}{c/n}.$$

Then the total travel time is

$$T = t_1 + t_2 = \frac{\sqrt{h_1^2 + x^2}}{c} + \frac{n\sqrt{h_2^2 + (L-x)^2}}{c}.$$

We are looking for an extremum, so

$$\frac{dT}{dx} = 0 = \frac{1}{2}\frac{2x}{c\sqrt{h_1^2 + x^2}} - \frac{1}{2}\frac{2n(L-x)}{c\sqrt{h_2^2 + (L-x)^2}}.$$

After canceling factors of c and 2, these fractions are precisely the sines of the angles

$$\Longrightarrow \boxed{\sin\theta_1 = n\sin\theta_2.} \qquad \text{Snell's law.}$$