Quantum Mechanics
Day 5

Today

I. Last Time

- We found an immensely useful formula for computing the variance and standard deviation

$$\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2 \qquad \text{and} \qquad \sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$$

- We thought carefully about probability densities and derived the main results for probability theory in the continuum formalism.
- We introduced Fermat's principle and used it to derive Snell's law.

II. Fermat's Principle in general

Of course, the index of refraction interface need not be as simple as it is in the Snell's law case, e.g. n could vary continuously $n = n(h)$. Indeed, this explains many desert mirages—the index of refraction of air is not exactly one and varies with temperature. This allows light leaving a palm tree to reach your eye along two different paths The lower ray looks as though it has been reflected off of water.

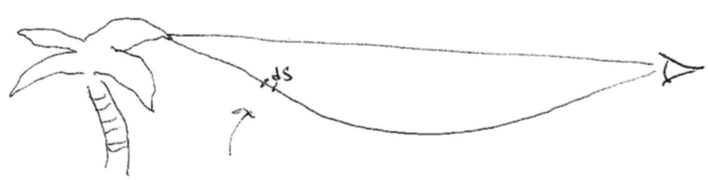


Figure 1: Two light rays leaving the same point on a palm tree can both reach your eye due to the varying index of refraction fo the heated air.

Let's set up the general case

$$T = \int_{t_0}^{t_1} dt = \frac{c}{c}\int_{t_0}^{t_1} \frac{dt}{ds} \cdot \left(\frac{ds}{dt}dt\right) = \frac{1}{c}\int_S^D \frac{c}{v}ds = \boxed{\frac{1}{c}\int_S^D n(s)ds.}$$

where a small segment of the light ray has arclength ds. This shows that the total travel time can be expressed directly in terms of the varying index of refraction $n(s)$. In general, then, you try to find the path connecting S and D that extremizes this integral

$$\boxed{T = \frac{1}{c}\int_S^D n(s)ds.}$$

Now, we've all been taught that the wave theory of light won out, and so, these rays are fictitious. Really there are wave fronts and rays are just a nice guideline overlay. So, is Fermat's principle a mathematical curiosity? A "divine miracle"? No! Actually it follows most clearly from the wave theory.

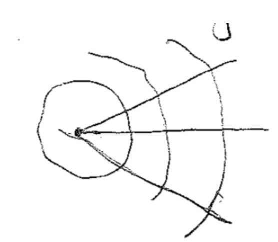


Figure 2: The wavefronts of a point source with rays traced as a guide.

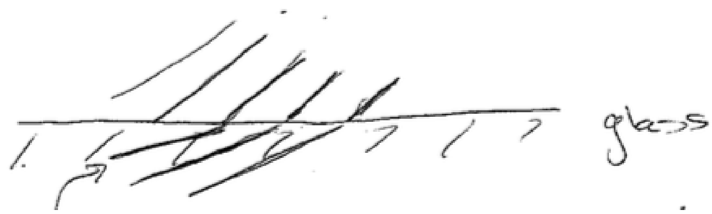
III. Huygens' principle

It is not our goal here to develop in full the wave theory of light, but let's pull in one of its beautiful principles.

If you're given a wave front, say of a plane wave, how do you construct the next one in the train? C. Huygens gave an answer: break the front into a (large) collection of point sources, then the next front is give by the superposition of spherical waves emitting from these sources. In particular, the next wavefront is the geometrical envelop of these spherical wavelets. Notice the clean role of locality in this construction.

Figure 3: The wavefronts of a point source with rays traced as a guide.
Figure 4: example caption

The Fermat principle follows simply from Huygens' principle. Let's take an example again. Because the local speed changes at



the glass interface, the wave fronts bend. This is due to wavelets in the air rapidly catching up to those in the glass. A useful analogy here is to a marching band marching across a partially dry and partially muddy field. Marchers entering the mud slow and the other marchers cover more ground, causing the wave front to ben.

IV. Foreshadowing

We turn now to discuss Hamilton's remarkable observation that there is a formulation of mechanics quite similar to Fermat's principle of optics.

V. The Opto-Mechanical Analogy

Hamilton noticed a striking correspondence. If we think of mechanics as the pursuit of finding trajectories then it is quite similar to finding optical rays.

If energy is conserved what determines how quickly a particle moves through space? Well,

$$\frac{p^2}{2m} + V = E,$$

so

$$p = \sqrt{2m(E - V)}.$$

We see that essentially the potential energy acts like an index of refraction for the particle. This led Hamilton to introduce

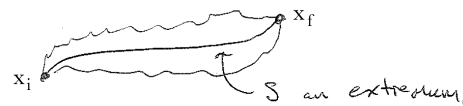


Figure 5: Amongst all trajectories connecting x_i and x_f the physical trajectory is the one that extremizes the action S.

Hamilton's Principle: A particle travels the path between two fixed points in space that extremizes the 'action'

$$S = \int_{x_i}^{x_f} p dx.$$

Notice that

$$\begin{aligned}
[S] \equiv & \text{ action} \\
= & [p][x] \\
= & \text{ angular momentum } (\vec{L} = \vec{r} \times \vec{p}) \\
= & \text{ energy} \cdot \text{time} \qquad \left(\frac{1}{2}(mv)v \cdot t\right) \\
= & [\hbar].
\end{aligned}$$

Action, while unfamiliar, is a remarkably versatile unit. Because action is such a useful unit, our last line here defines a new constant. All we know about this constant is that it has units of action and that we are going to call it 'hbar'.

As an example of Hamilton's principle, consider a free particle, that is, a particle not subject to any potential, traveling from (x_i, y_i) to (x_f, y_f) via an intermediate point (x, y). Then, let

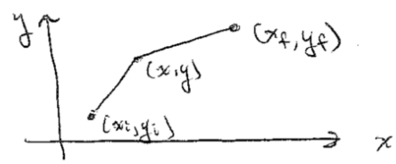


Figure 6: What intermediate point (x, y) gives the physical trajectory that connects given endpoints?

$$\begin{aligned}
S_1 &= \sqrt{2mE}\sqrt{(x - x_i)^2 + (y - y_i)^2} \equiv \sqrt{2mE} s_1 \quad \text{and} \\
S_1 &= \sqrt{2mE}\sqrt{(x_f - x)^2 + (y_f - y)^2} \equiv \sqrt{2mE} s_2.
\end{aligned}$$

Here s_1 and s_2 are just convenient shorthands for the two big square rooted expressions. The total action is

$$S = S_1 + S_2.$$

We extremize, first x,

$$\frac{\partial S}{\partial x} = 0 \quad \Longrightarrow \quad \sqrt{2mE}\left(\frac{(x - x_i)}{s_1} - \frac{(x_f - x)}{s_2}\right) = 0,$$

then y,

$$\frac{\partial S}{\partial y} = 0 \quad \Longrightarrow \quad \sqrt{2mE}\left(\frac{(y - y_i)}{s_1} - \frac{(y_f - y)}{s_2}\right) = 0.$$

These two conditions are equivalent to

$$\frac{x - x_i}{x_f - x} = \frac{y - y_i}{y_f - y} \qquad \text{or} \qquad \frac{y_f - y}{x_f - x} = \frac{y - y_i}{x - x_i}.$$

The slopes of the two segments are equal! The physical trajectory is a straight line. A result that is completely consistent with what we know about free particles.

Huygens' principle leads us to ask whether a wave explanation can also be given for Hamilton's principle?