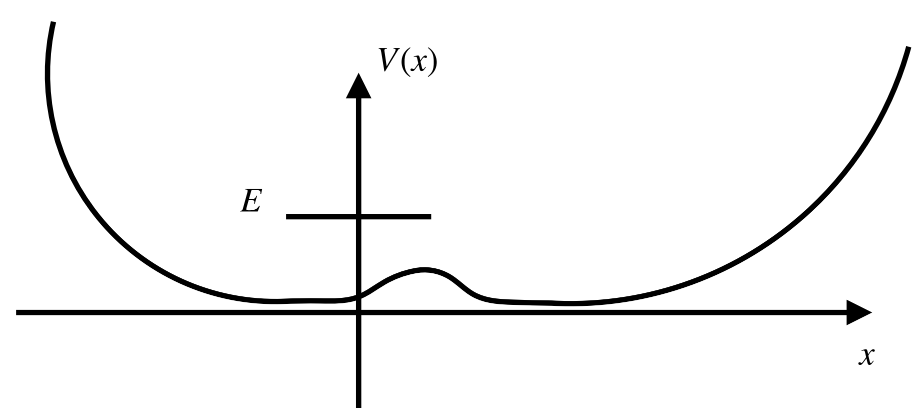
Today

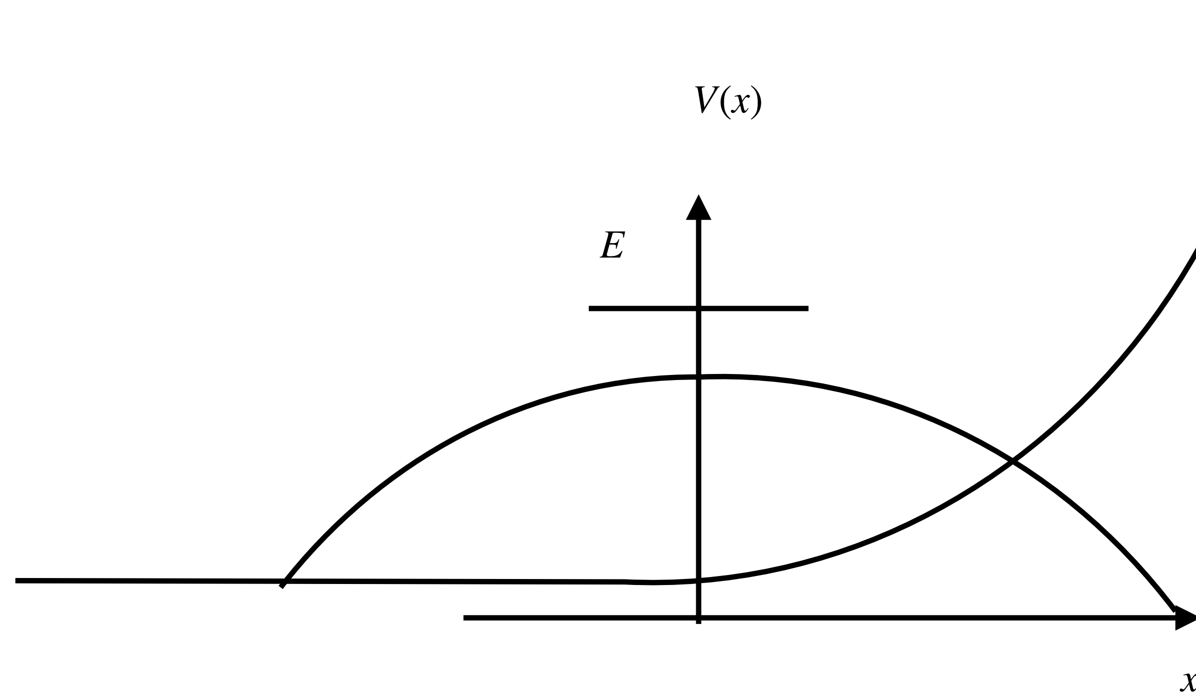
I. Last Time
II. The Delta Function Well and Bound States
III. The Delta Function Well and Scattering States

I. Bohr-Einstein debate on Energy-Time uncertainty

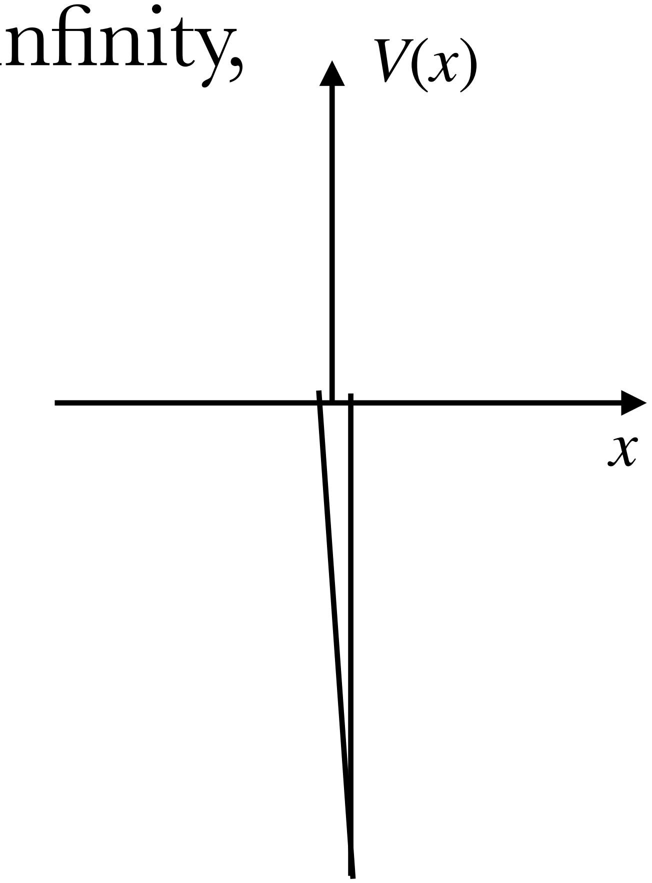
*Bound states and scattering states



Bound state: $E < V(-\infty)$ **and** $V(\infty)$



For scattering the energy of the state is either greater than the potential at infinity or at -infinity, or both.



Suppose $V(x) = -\alpha\delta(x) \quad \text{with} \quad \alpha > 0$

The Schrodinger equation gives

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\delta(x)\psi(x) = E\psi(x)$$

a. Bound states $E < V(\pm\infty) = 0$ b. Scattering states $E > 0$

a. (1) In $x < 0$, $\delta(x) = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \Longrightarrow \quad \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = k^2\psi$$

$$\psi(x) = Ae^{-kx} + Be^{kx}, \quad x < 0 \quad \text{With} \quad k = \frac{\sqrt{-2mE}}{\hbar}$$

a. (1) In $x > 0$, $\delta(x) = 0$, same

$$\psi(x) = Fe^{-kx} + Ge^{kx} \quad x > 0$$

Impose boundary conditions: as position goes to infinity, we want a finite wave function. So, set A=0 for x<0 and G=0 for x>0.

There are two boundary conditions we should also consider at x=0…

$\psi(x)$ continuous

$\dfrac{d\psi(x)}{dx}$ continuous, except at a pt where V=∞

Continuity at x=0 gives

$$Be^0 = Fe^0 \quad \Longrightarrow \quad B = F\,.$$

So,

$$\psi(x) = \begin{cases} Be^{kx} & x < 0 \\ Be^{-kx} & x > 0 \end{cases} = Be^{-k|x|}$$

Consider,

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}dx - \alpha\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)dx = E\int_{-\epsilon}^{\epsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}dx - \alpha\psi(0) = 0$$

Consider,

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{d^2\psi}{dx^2}dx - \alpha\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)dx = E\int_{-\epsilon}^{\epsilon}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\left(\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon}\right) - \alpha\psi(0) = 0$$

Then

$$\Delta\left(\frac{d\psi}{dx}\right) = \alpha\psi(0)\left(-\frac{2m}{\hbar^2}\right)$$

In our case, $\psi(0) = B,$ and so

$$\Delta\left(\frac{d\psi}{dx}\right) = \alpha B\left(-\frac{2m}{\hbar^2}\right)$$

as derived from the Sch. Eq.

$$\psi(x) = \begin{cases} Be^{kx} & x < 0 \\ Be^{-kx} & x > 0 \end{cases} = Be^{-k|x|}$$

$$\frac{d\psi(x)}{dx} = \begin{cases} kBe^{k(-\epsilon)} \to kB \\ -kBe^{-k\epsilon} \to -kB \end{cases}$$

Impose equality of the two results to find

$$\Delta\left(\frac{d\psi}{dx}\right) = -kB - kB = -2kB = \alpha B\left(-\frac{2m}{\hbar^2}\right)$$

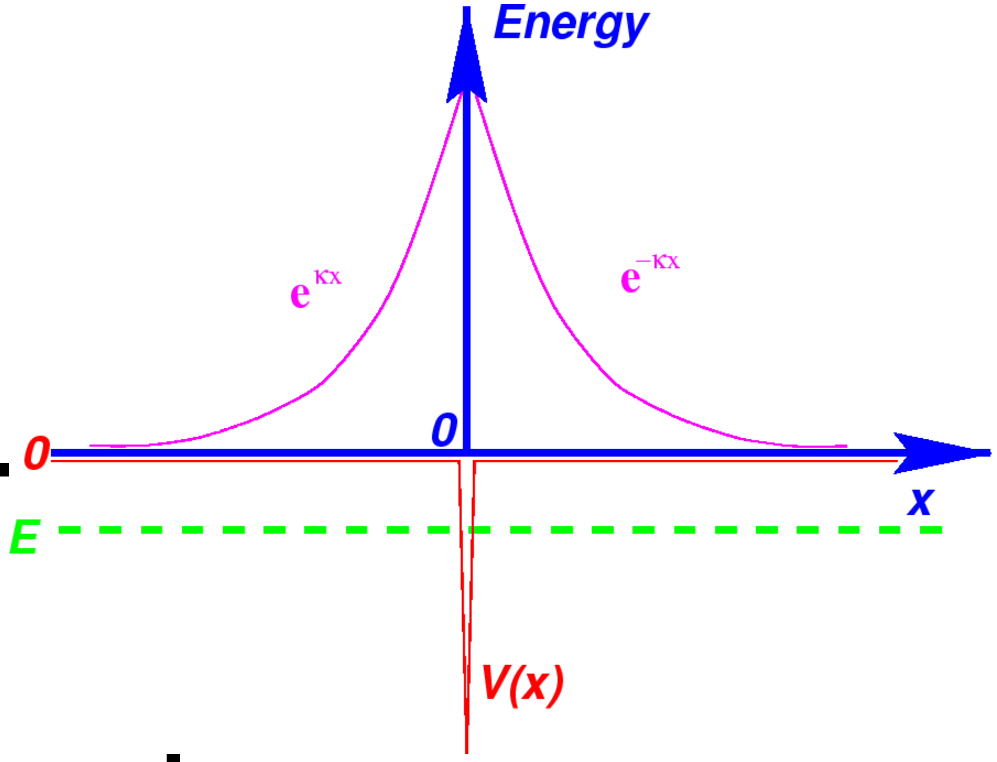
We've just learned that

$$k = \alpha\left(\frac{m}{\hbar^2}\right) = \frac{\sqrt{-2mE}}{\hbar}$$

$$E = -\frac{k^2\hbar^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$$

Finally we determine B by normalizing:

$$B = \sqrt{k}$$



$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha}{\hbar^2}|x|}, \quad \text{and} \quad E = -\frac{m\alpha^2}{2\hbar^2}\ .$$