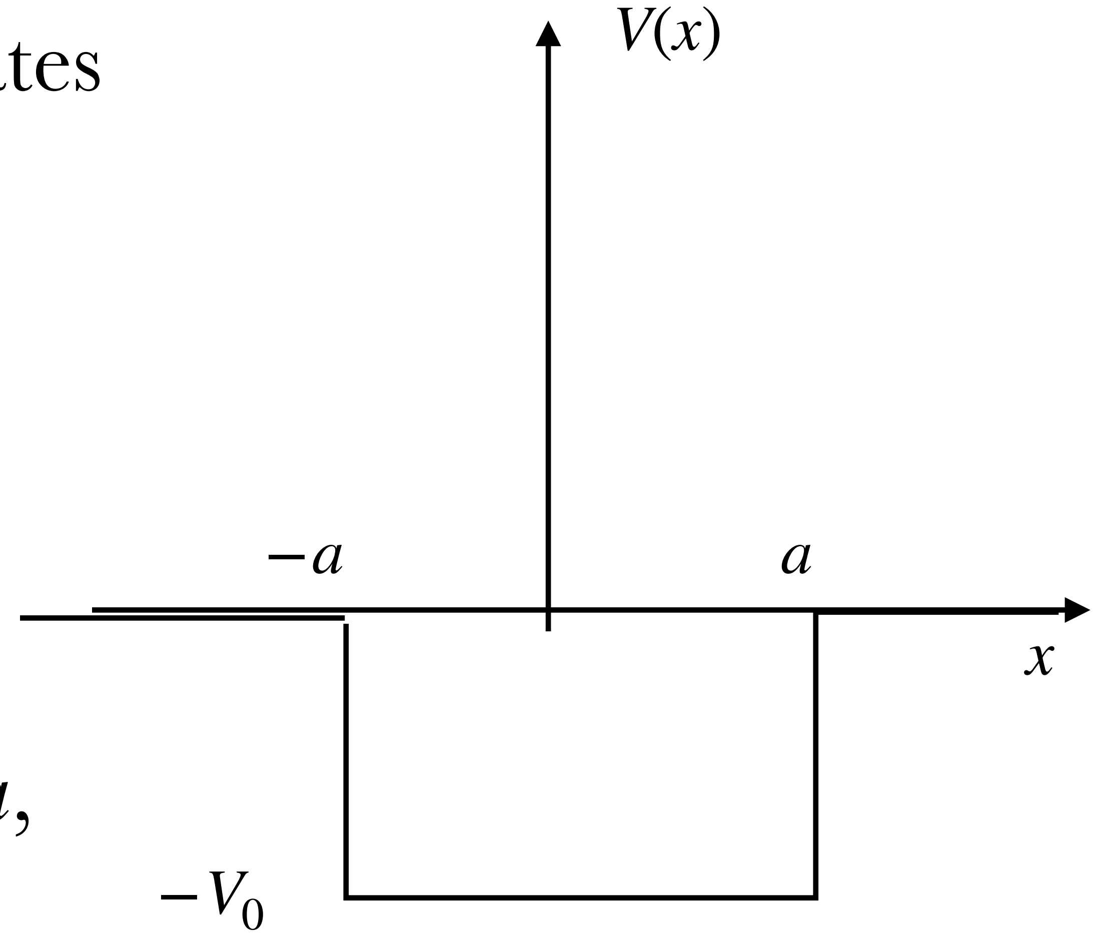
Today

I. Feedback Summary
II. Last Time
III. Finite Square Well: Bound States

I. Last time

*Scattering off of a delta function well
*Wave functions aren't normalizable in general for scattering→ Transmission and Reflection coefficients
*T and R fraction of particles transmitted or reflected respectively. (relative probabilities).
*Conventional choice: sent particles in from negative ∞, and set $G = 0$ hand.
*Intro'd tunneling by noticing a delta barrier has non-zero T.

III. Finite square well: bound states



One more 1D example:

$$V(x) = \begin{cases} -V_0 & -a \le x \le a \\ 0 & |x| > a \end{cases}$$

Bound states: $E < 0$. For $x < -a$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \text{or} \quad \frac{d^2\psi}{dx^2} = k^2\psi$$

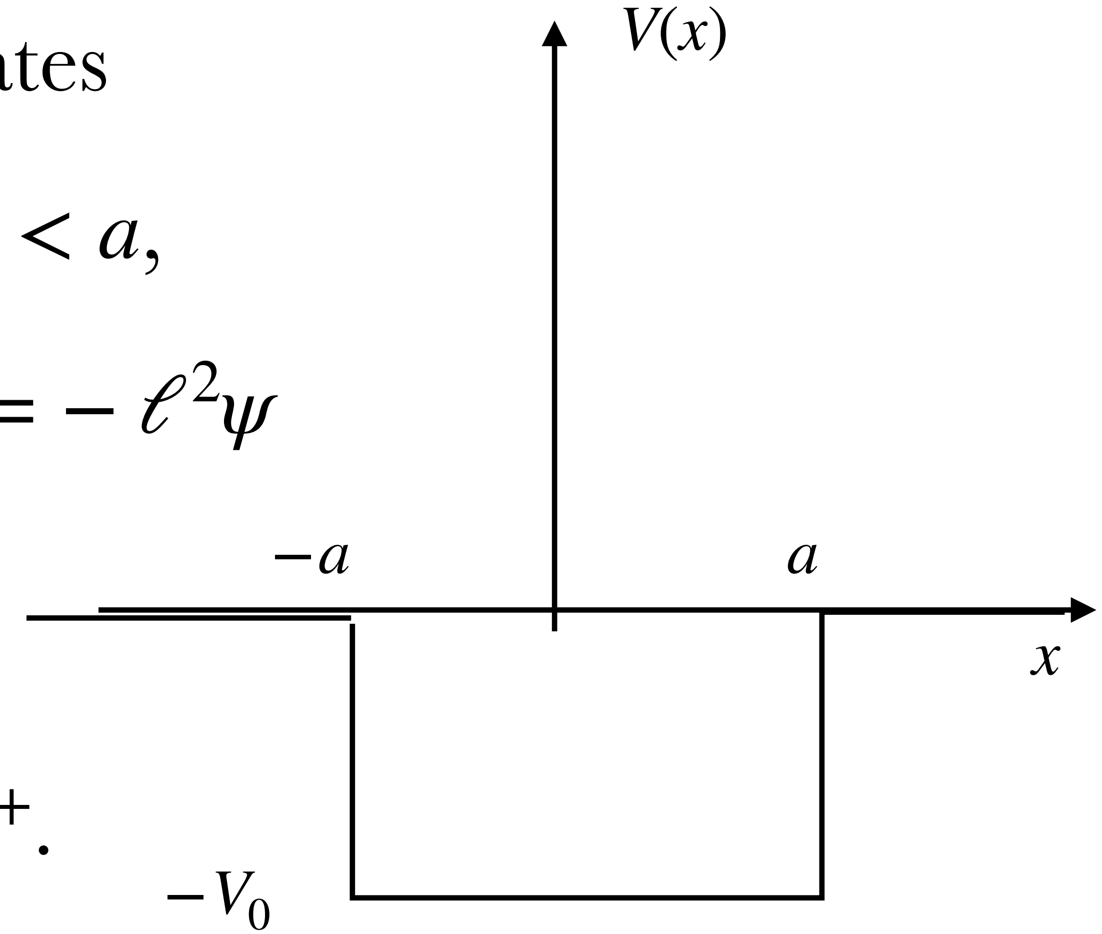
With $\quad k = \dfrac{\sqrt{-2mE}}{\hbar}$ (recall $E < 0$)

Then, $\psi(x) = Ae^{-kx} + Be^{kx}$

But as $x \to -\infty$, e^{-kx} blows up, so A=0 and

Then, $\psi(x) = Be^{kx} \qquad x < -a.$

III. Finite square well: bound states



Bound states: $E < 0$. For $-a < x < a$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \quad \text{or} \quad \frac{d^2\psi}{dx^2} = -\ell^2\psi$$

With $\quad \ell = \dfrac{\sqrt{2m(E+V_0)}}{\hbar}$

(recall $E > V_{\text{min}} = -V_0$), $\ell \in \mathbb{R}^+$.

Then, $\psi(x) = C\sin(\ell x) + D\cos(\ell x) \qquad -a < x < a$

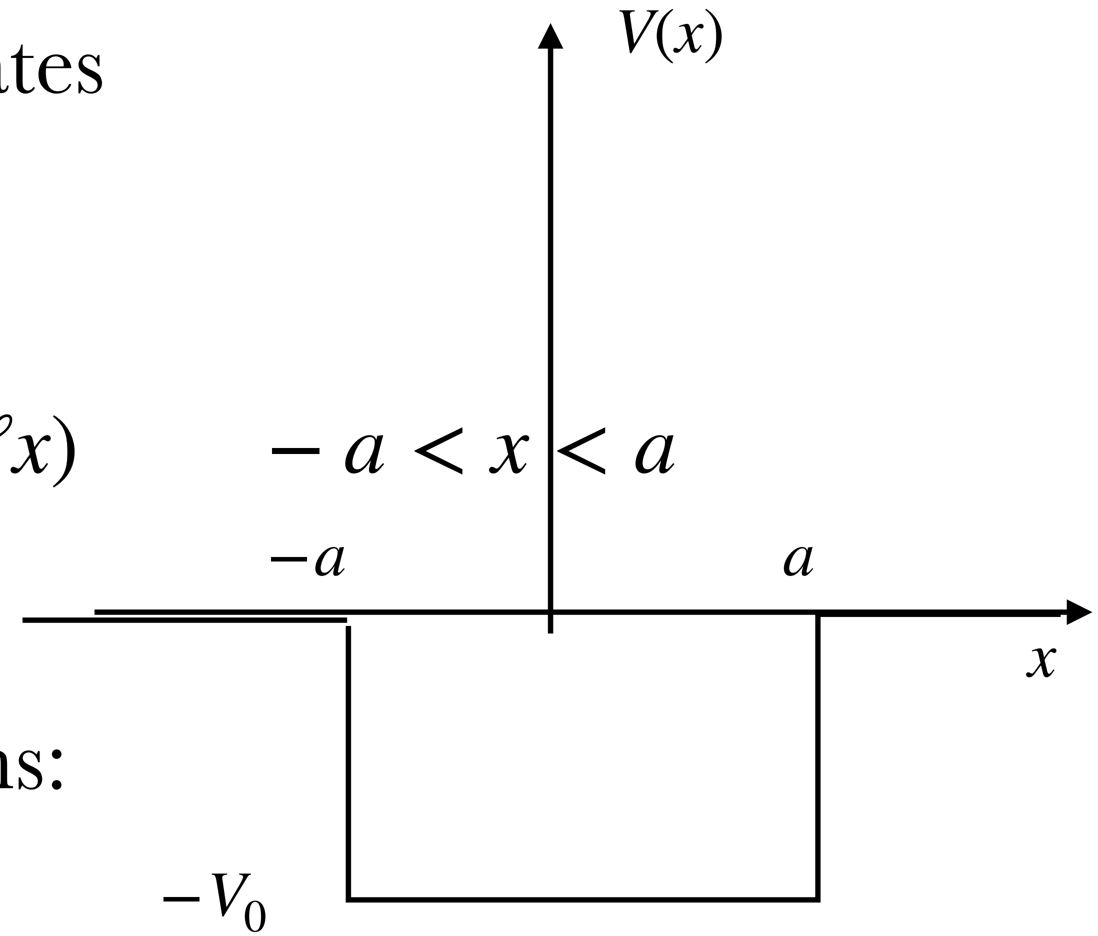
Bound states: $E < 0$. For $x > a$,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \text{or} \quad \frac{d^2\psi}{dx^2} = k^2\psi$$

With $\quad k = \dfrac{\sqrt{-2mE}}{\hbar}$ (recall $E < 0$)

Then, $\psi(x) = Fe^{-kx} + Ge^{kx} \quad x > a$, but as $x \to \infty$, e^{kx} blows up, so G=0.

III. Finite square well: bound states



Then, $\psi(x) = Be^{kx} \qquad x < -a.$

Then, $\psi(x) = C\sin(\ell x) + D\cos(\ell x) \qquad -a < x < a$

Then, $\psi(x) = Fe^{-kx} \quad x > a.$

Next impose boundary conditions:

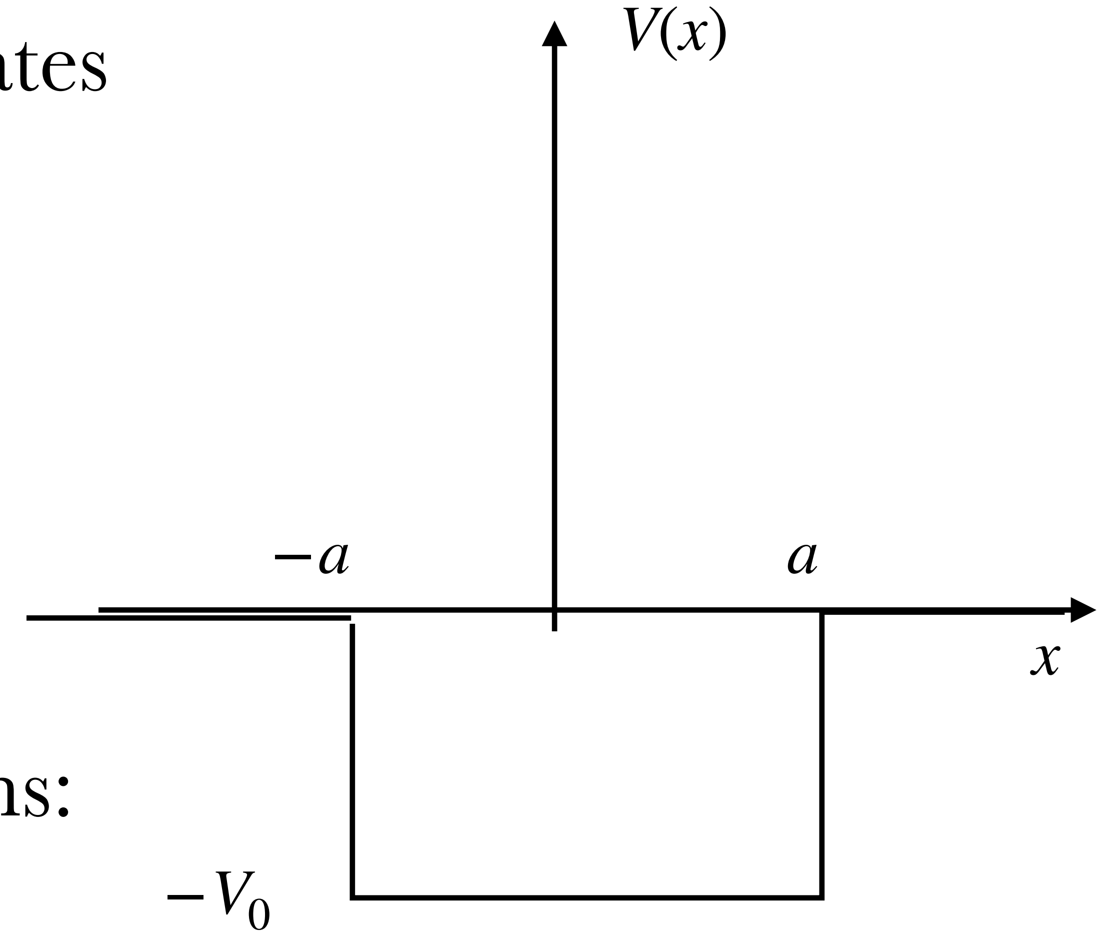
ψ and $\dfrac{d\psi}{dx}$ are continuous.

But, since the potential is even, we can look for even or odd solutions —a general solution is built out of these as a superposition.

Let's do the even one:

$$\psi(x) = \begin{cases} Fe^{-kx} & x > a \\ D\cos(\ell x) & 0 < x < a \\ \psi(-x) & x < 0 \end{cases}$$

III. Finite square well: bound states



Let's do the even one:

$$\psi(x) = \begin{cases} Fe^{-kx} & x \geq a \\ D\cos(\ell x) & 0 \leq x \leq a \\ \psi(-x) & x \leq 0 \end{cases}$$

Next impose boundary conditions:

ψ and $\dfrac{d\psi}{dx}$ are continuous.

At $x = a$,

$Fe^{-ka} = D\cos(\ell a)$

$-kFe^{-ka} = -\ell D\sin(\ell a)$

Dividing these two equations

$k = \ell\tan(\ell a)$

Let's simplify this a bit $z = \ell a$, and $z_0 = \dfrac{a}{\hbar}\sqrt{2mV_0}$

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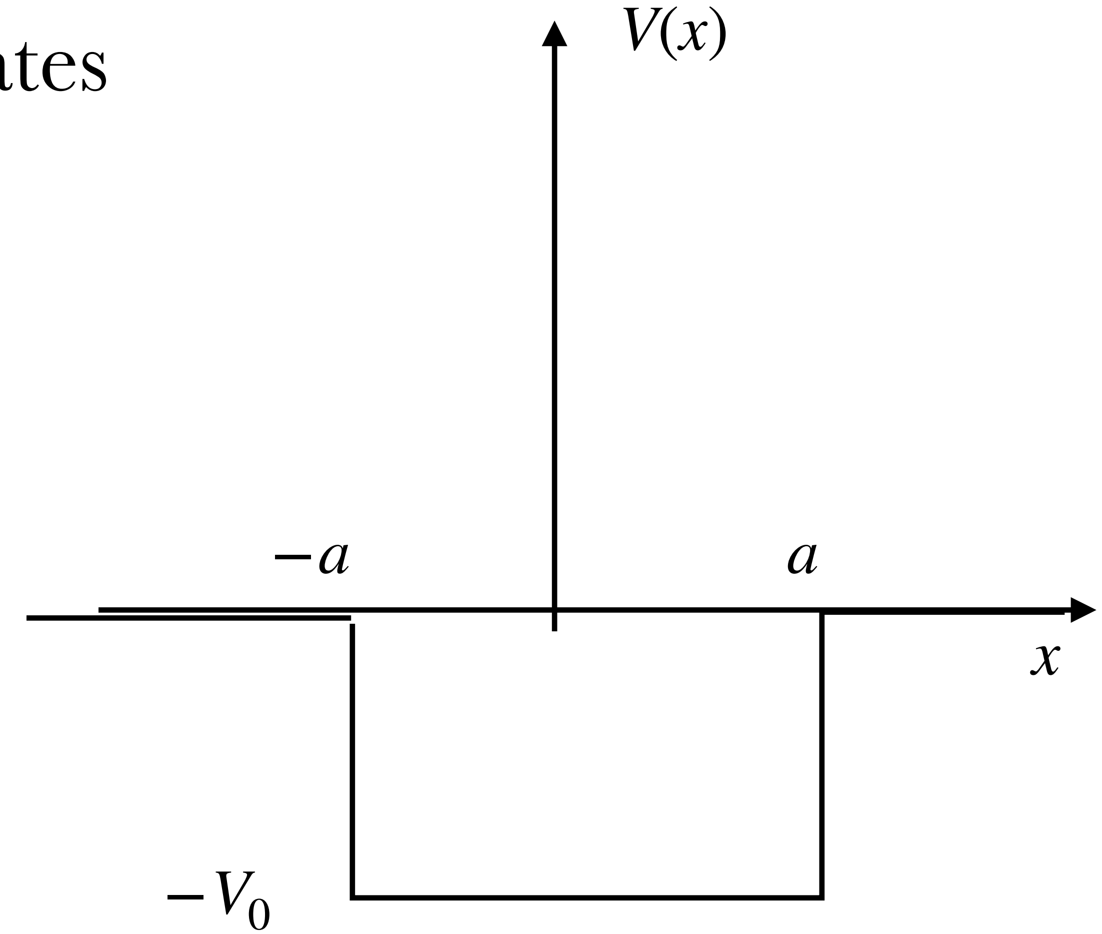

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Dividing these two equations

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Let's simplify this a bit $z = \ell a$, and $z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$. Now,

$k^2 + \ell^2 = \frac{2mV_0}{\hbar^2}$ and then $k = \sqrt{\frac{2mV_0}{\hbar^2} - \ell^2}$ or

$ka = \sqrt{z_0^2 - z^2}$. This allows us to write

$ak = a\ell\tan(\ell a)$ or $\sqrt{z_0^2 - z^2} = z\tan(z)$ or $\tan(z) = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$

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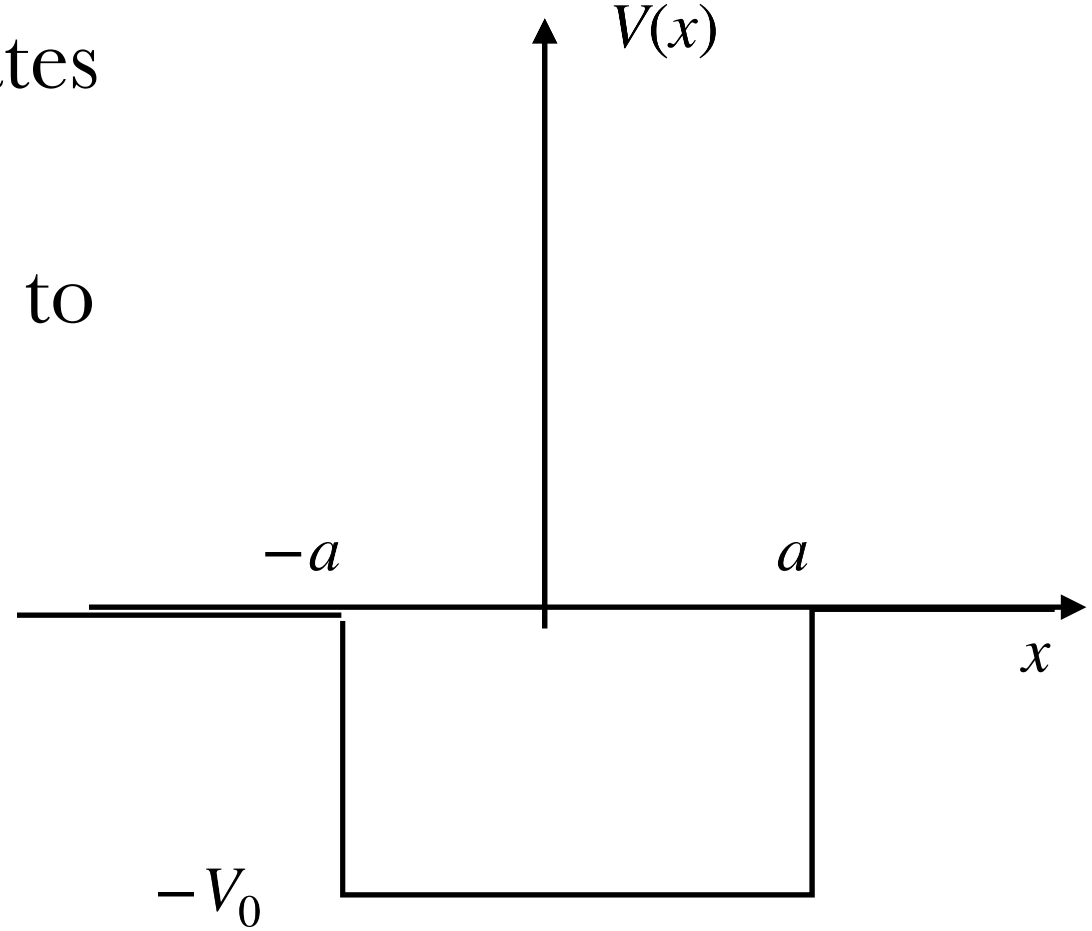
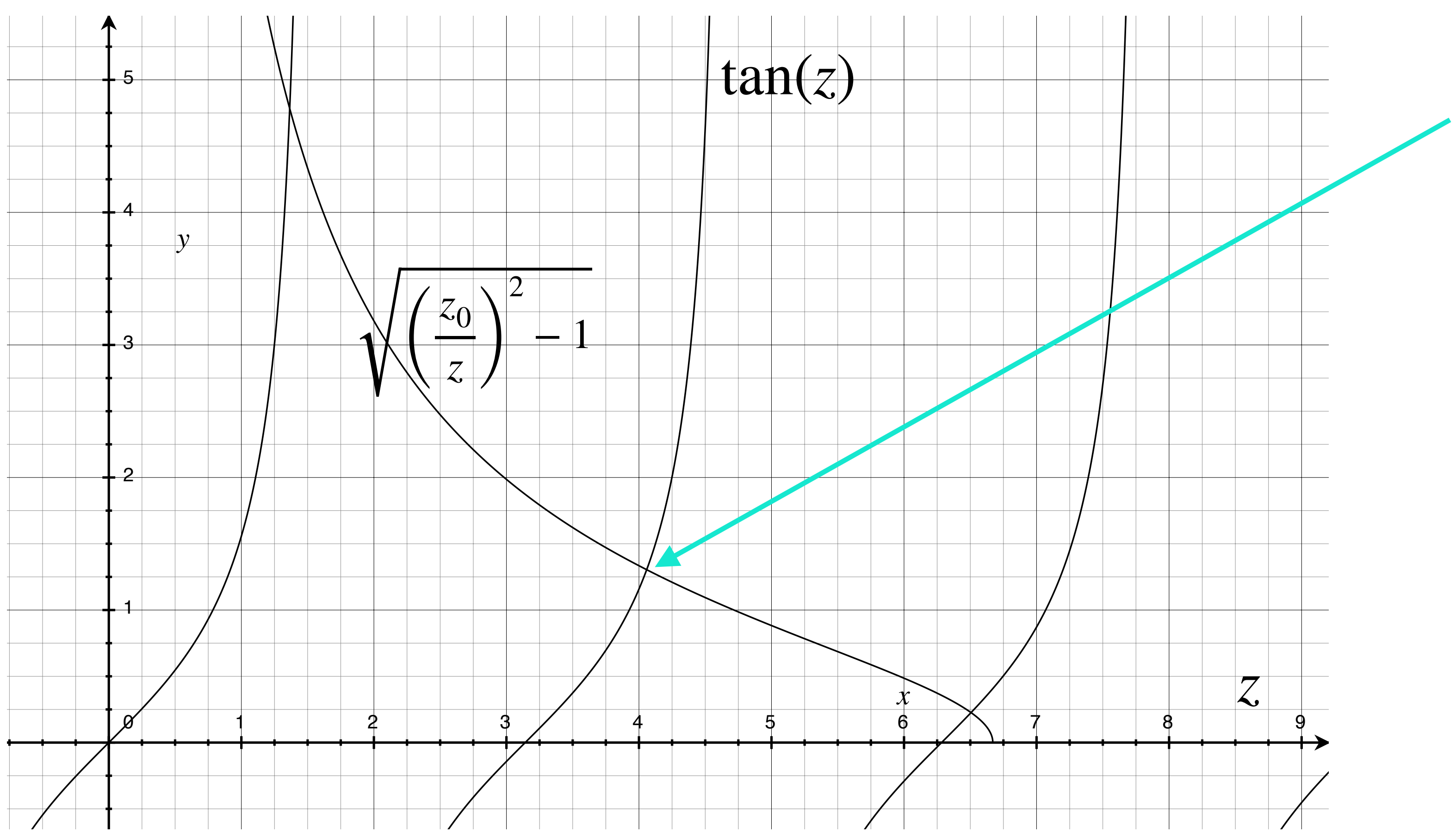


Our boundary conditions led us to

$$\tan(z) = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$



Intersections give the allowed z, which give allowed ℓ, which give allowed E.