Today

I. Last Time
II. Two-Dimensional Quantum Theory
III. 2D Infinite Square Well
IV. The Circular Billiard and Angular Momentum

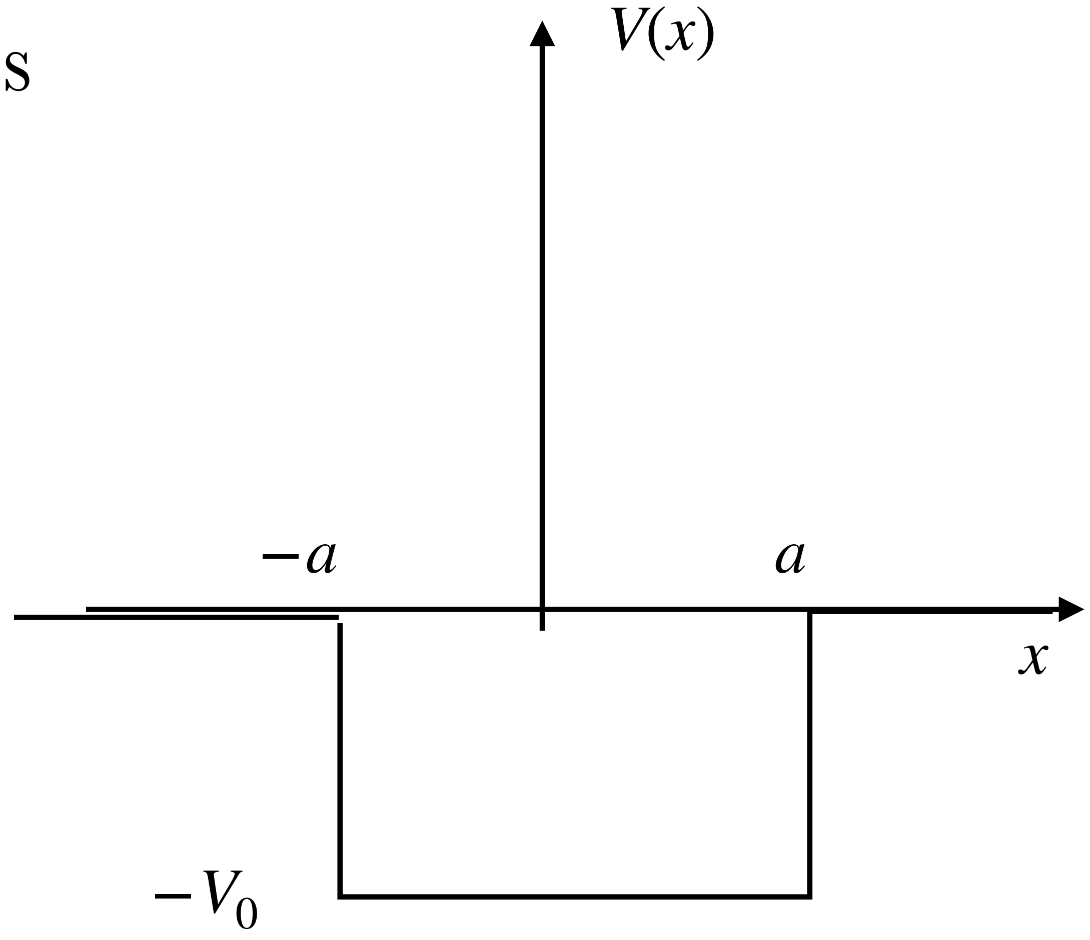
I. Last time

* Worked out the even solutions for the bound states of the finite square well.
* To find the energies we had to solve a transcendental equation, which we did numerically, by plotting.

I. Finite square well: bound states

One more 1D example:

$$V(x) = \begin{cases} -V_0 & -a \le x \le a \\ 0 & |x| > a \end{cases}$$

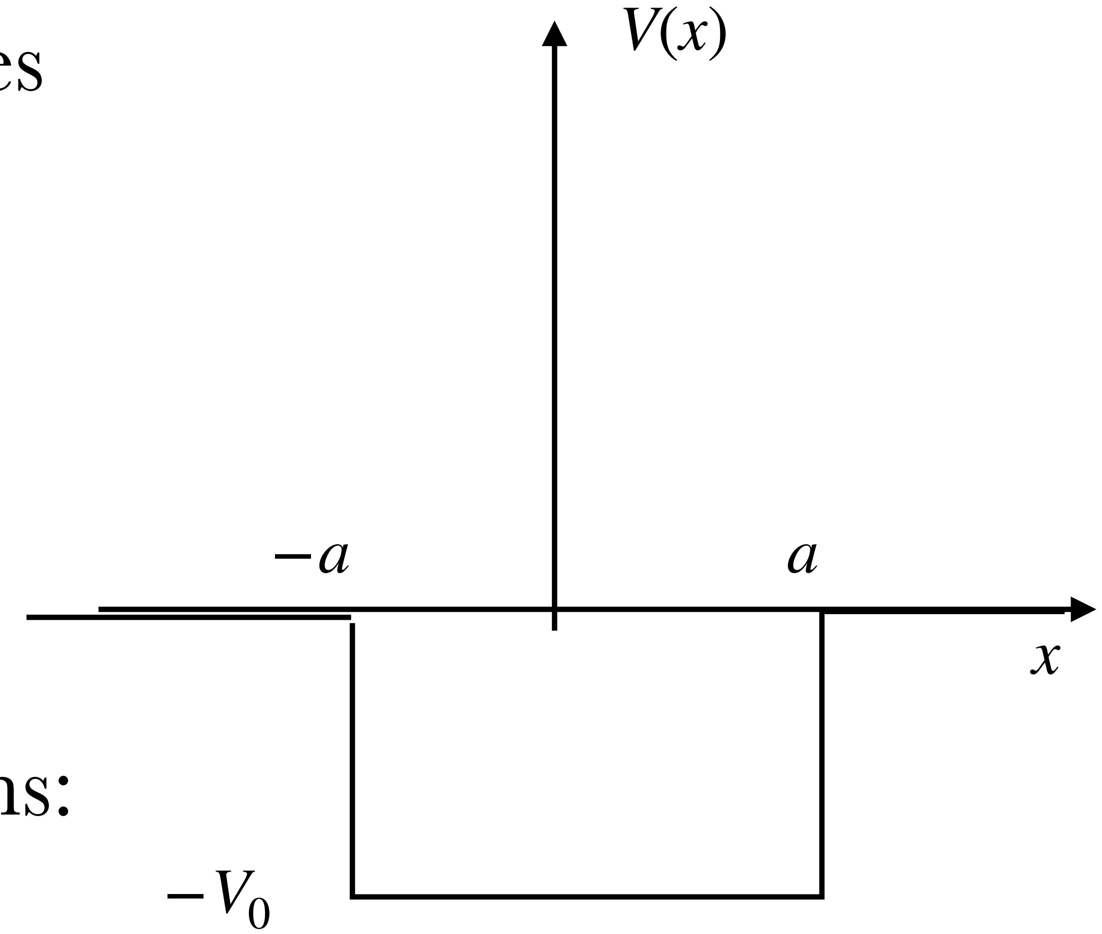


But, since the potential is even, we can look for even or odd solutions —a general solution is built out of these as a superposition.

Let's do the even one:

$$\psi(x) = \begin{cases} Fe^{-kx} & x > a \\ D\cos(\ell x) & 0 < x < a \\ \psi(-x) & x < 0 \end{cases}$$

I. Finite square well: bound states



Let's do the even one:

$$\psi(x) = \begin{cases} Fe^{-kx} & x \geq a \\ D\cos(\ell x) & 0 \leq x \leq a \\ \psi(-x) & x \leq 0 \end{cases}$$

Next impose boundary conditions:

ψ and $\frac{d\psi}{dx}$ are continuous.

At $x = a$,

$Fe^{-ka} = D\cos(\ell a)$

$-kFe^{-ka} = -\ell D\sin(\ell a)$

Dividing these two equations

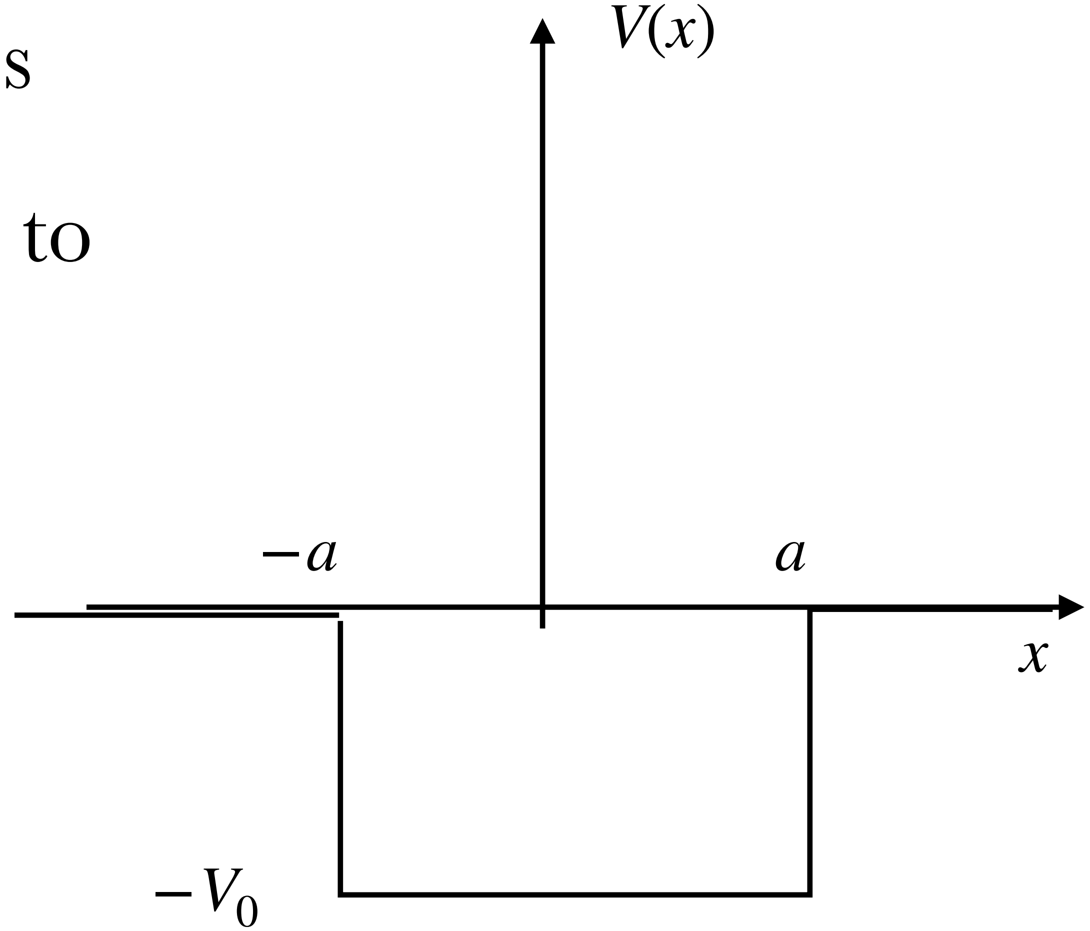
$k = \ell\tan(\ell a)$

Let's simplify this a bit $z = \ell a$, and $z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$

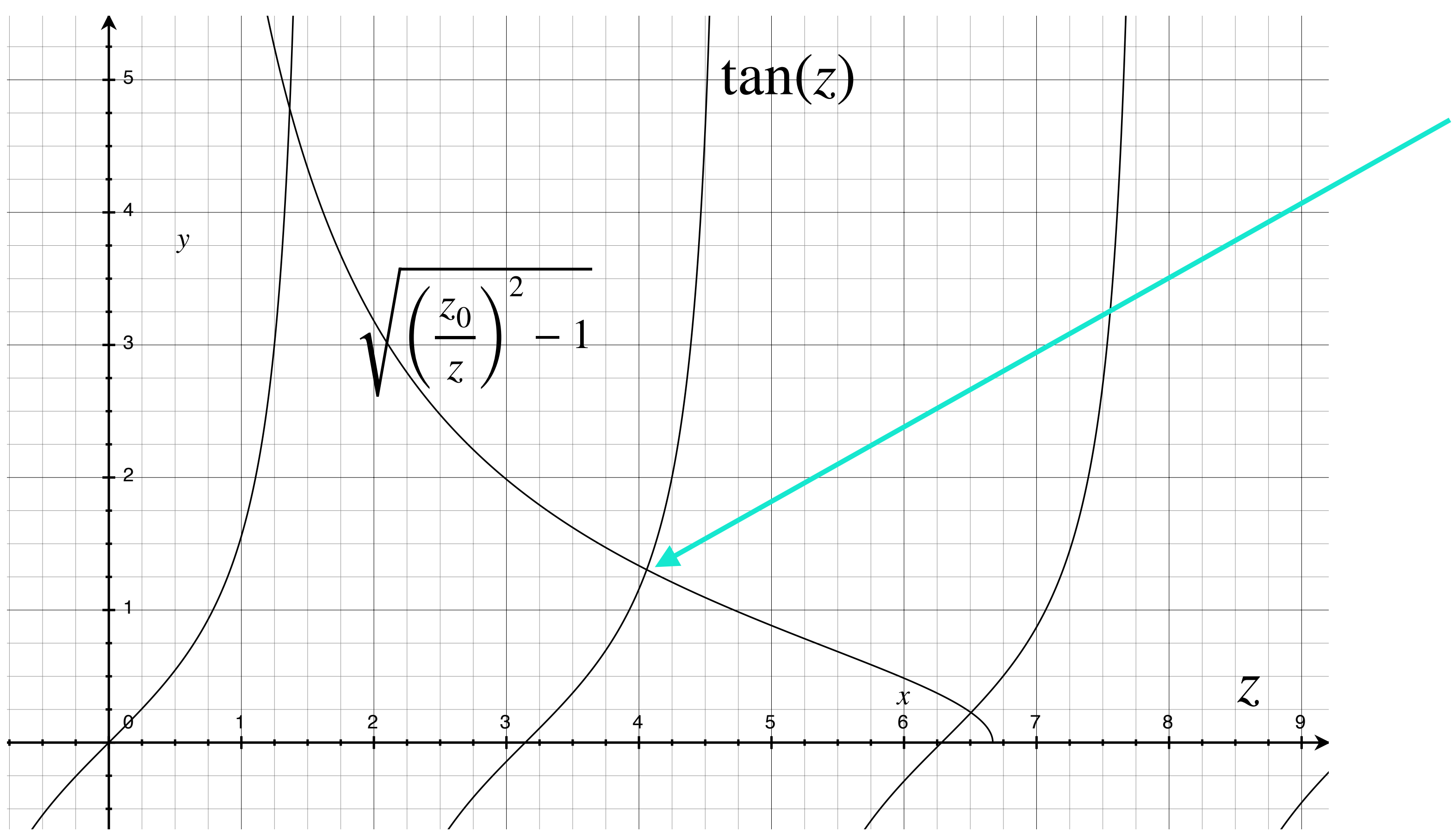
I. Finite square well: bound states

Our boundary conditions led us to the transcendental relation

$$\tan(z) = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$

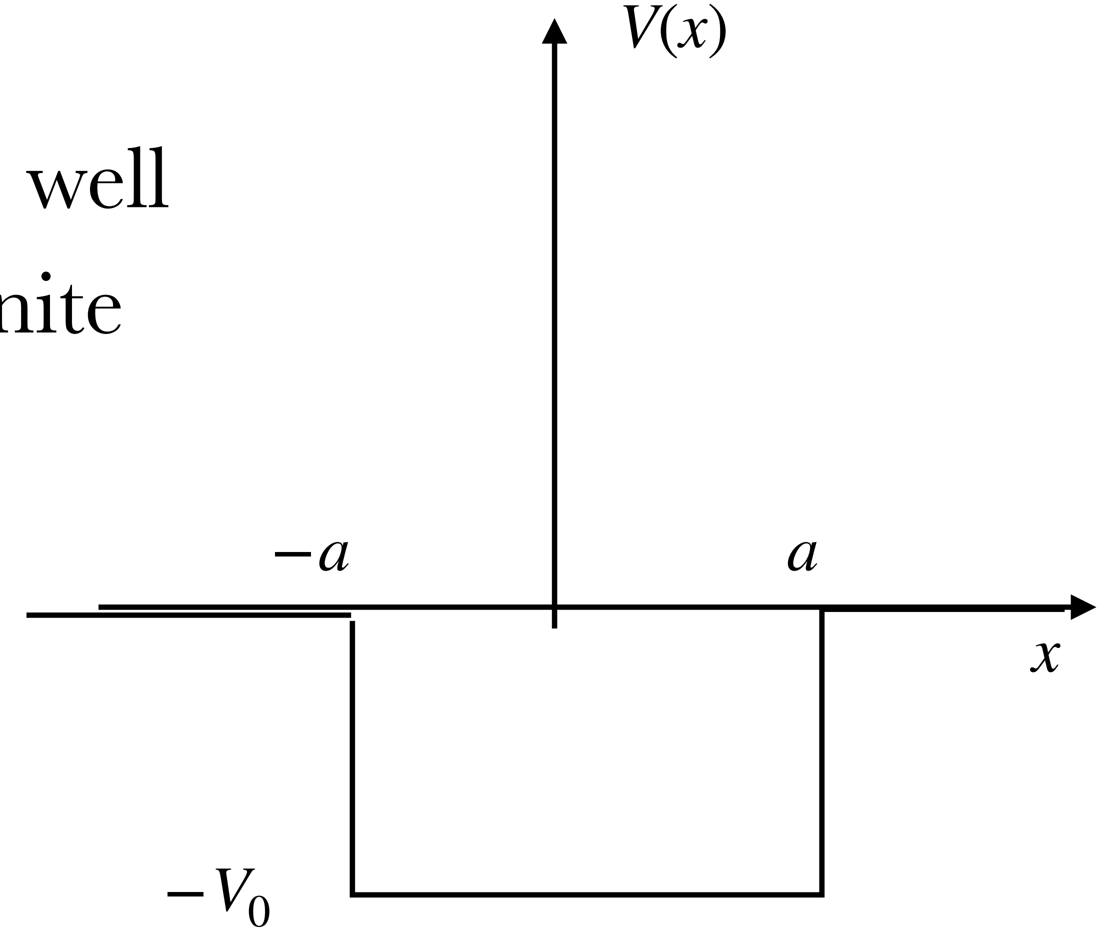


$z_0 = 6.67$



Intersections give the allowed z, which give allowed ℓ, which give allowed E.

Limiting cases: For a wide, deep well this should be related to the infinite square well.

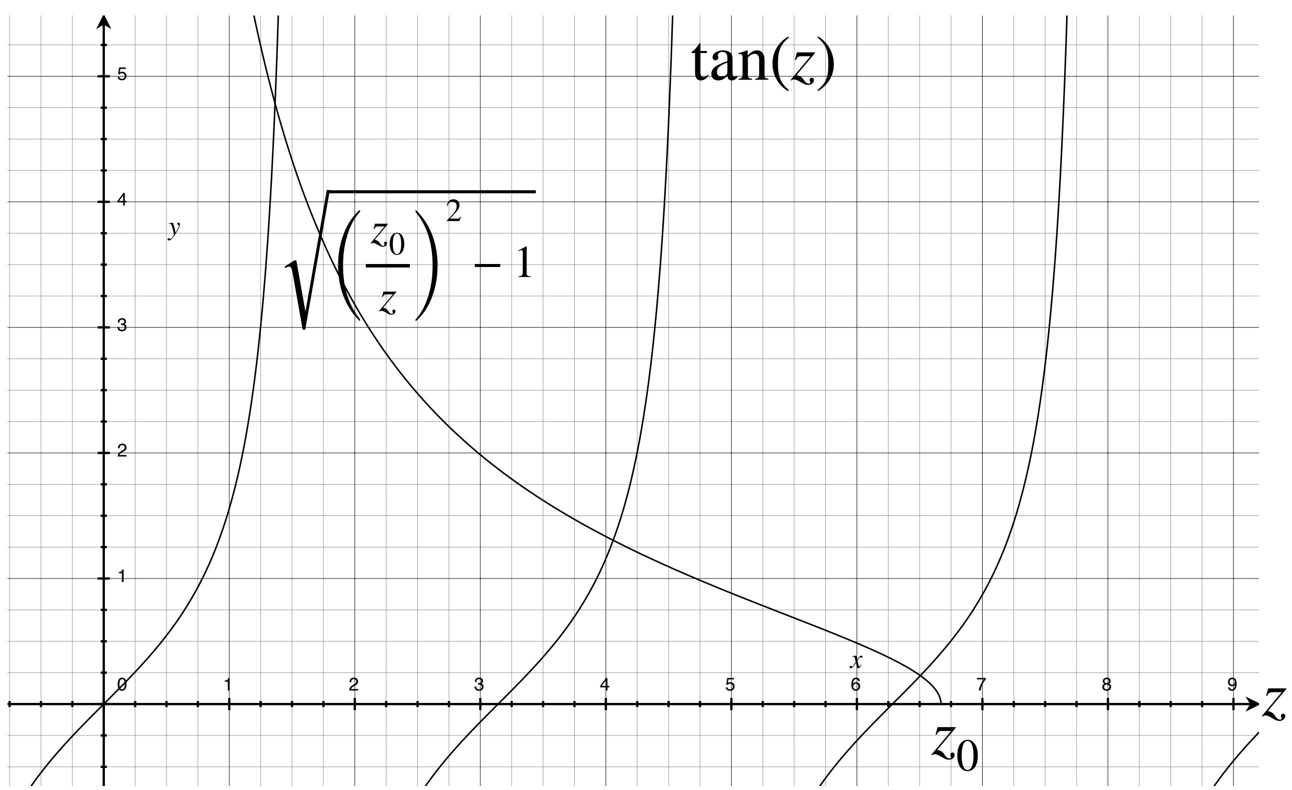


$$z_{int} \approx z_n = \frac{n\pi}{2}$$

then, the allowed energies are:

$$z_n = \ell_n a = \frac{a\sqrt{2m(E_n + V_0)}}{\hbar} = \frac{n\pi}{2}$$

$z_0 = 6.67$



Or

$$E_n + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

These are the square well solutions above V_0, but only works for finitely many n.

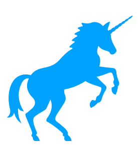
In the narrow, shallow limit there's only one solution, but it's there!

II. 2D Quantum Theory

We've been generating more and more examples of quantum systems by changing the shape of the potential. There's another way to do it. That is, we can change the dimensions of space. Let's consider 2D.

What becomes of the Schroedinger equation?
We just add an argument to Ψ, that is

$\Psi(x, t) \to \Psi(x, y, t)$

[ (important) Aside on notation: We're lazy and write

2D: $\Psi(x, y, t) = \Psi(\vec{r}, t)$

2D: $\vec{r} = (x, y)$

3D: $\vec{r} = (x, y, z)$

Notice that this notation, while odd, is consistent:

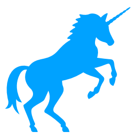
2D: $r = |\vec{r}| = \sqrt{x^2 + y^2}$ (polar radius)

3D: $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ (spherical radius)

II. 2D Quantum Theory

We just add an argument to Ψ, that is

$\Psi(x, t) \rightarrow \Psi(x, y, t)$

[ (important) Aside on notation: We're lazy and write

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But, of course, $r \neq \vec{r}$. For integration it's convenient to intro:

2D: $d^2\vec{r} = dxdy$ $\qquad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

3D: $d^3\vec{r} = dxdydz = r^2 \sin\theta dr d\theta d\phi$ $\qquad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$]

Schroedinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right) + V(x,y)\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

What kind of differential equation is this? 2nd order, partial differential equation. (2nd order PDE.) We solve PDEs by separation of variables.

Let's do it in stages, let's separate t first. We'll get a time-indep Sch.

$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + V\psi = E\psi$, where $\psi = \psi(x,y)$. There's also a wiggle factor: $\varphi(t) = e^{-iEt/\hbar}$. Let's separate the final variables, x and y:

$\psi(x,y) = X(x)Y(y)$ separation ansatz

$$-\frac{\hbar^2}{2m}\left(Y(y)\frac{d^2X}{\partial x^2} + X(x)\frac{\partial^2 Y}{\partial y^2}\right) + V\psi = EXY$$

Let's separate the final variables, x and y:

$\psi(x, y) = X(x)Y(y)$ separation ansatz

$$-\frac{\hbar^2}{2m}\left(Y(y)\frac{d^2X}{dx^2} + X(x)\frac{d^2Y}{dy^2}\right) + VXY = EXY$$

Dividing through by XY gives

$$-\frac{\hbar^2}{2m}\left(\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}\right) + V_x(x) + V_y(y) = E$$

This leads us to two separated equations

$$-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} + V_x(x) = \text{const.} = E_x$$

$$-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} + V_y(y) = \text{const.} = E_y$$

And $E_x + E_y = E$. These are just two copies of the 1D Sch. Eq.

Then the full solutions is $\psi(x, y) = X(x)Y(y)$.