Today

I. Last Time
II. 2D Infinite Square Well
III. Angular Momentum

I. Last time

* Studied the 2D Sch. Eqn. Intro'd notation: $\vec{r} = (x, y)$. And $r_1 = x$ and $r_2 = y$.
* Separated variables and found $\psi(x, y) = X(x)Y(y)$ satisfied a Sch. Eqn. For each component:

$$-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} + V_x X = E_x X$$

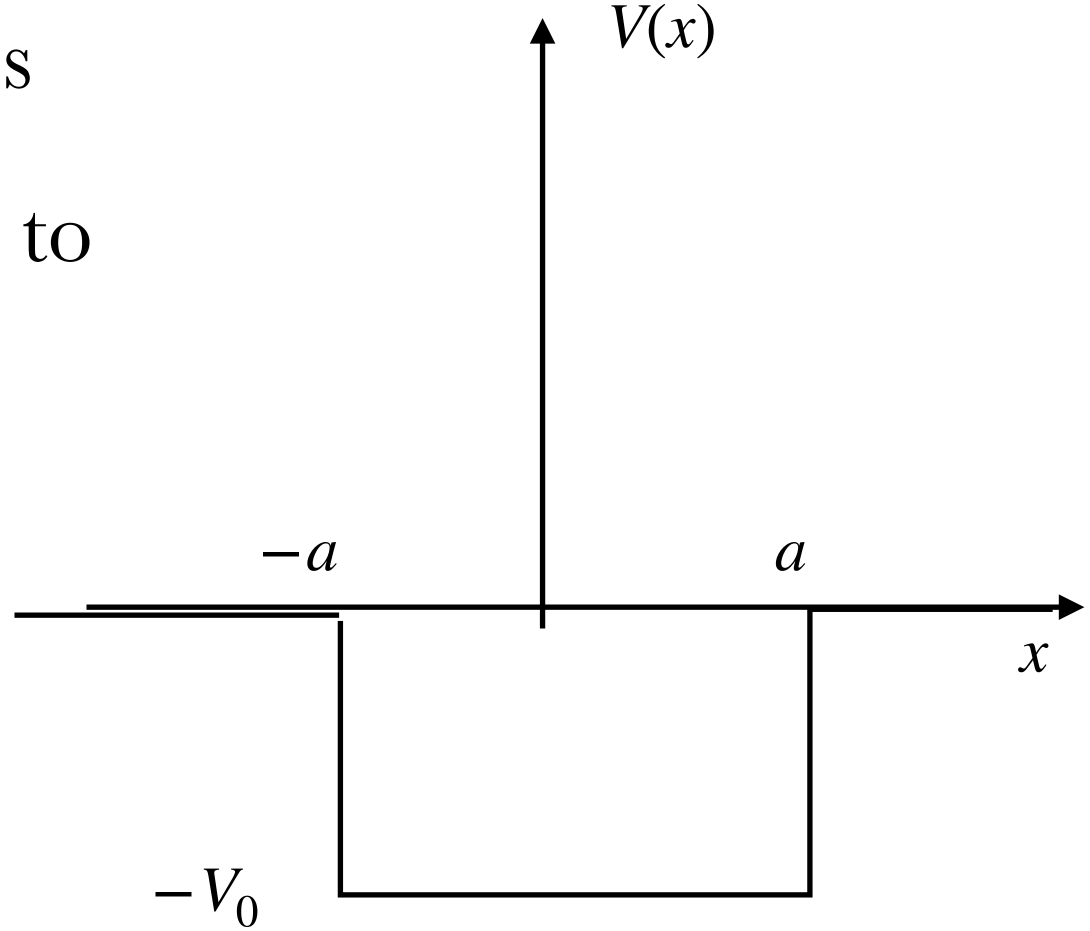
And similarly for Y, with $E_x + E_y = E$.

*Studied the wide and deep and narrow and shallow limits of the finite square well.

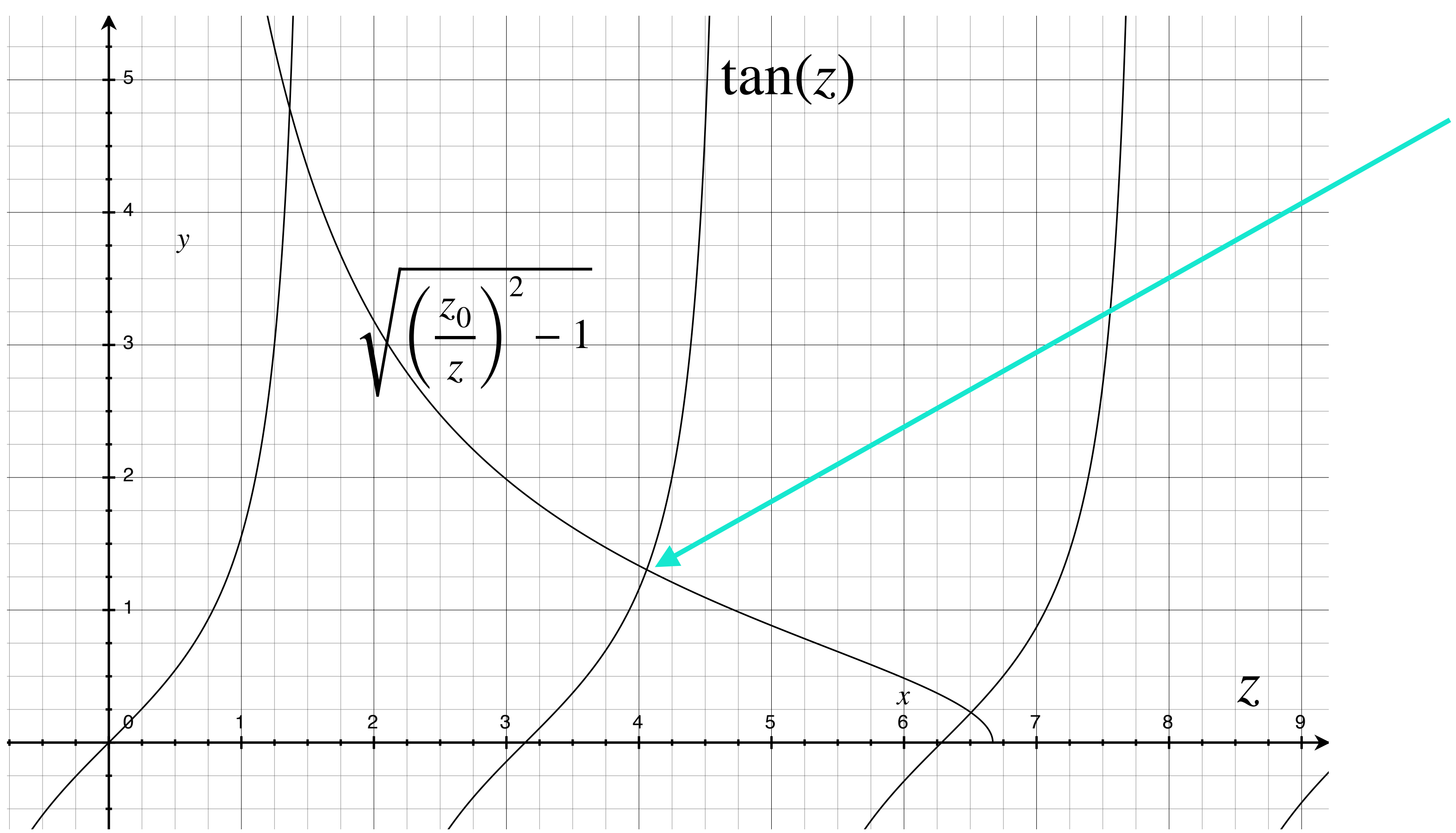
I. Finite square well: bound states

Our boundary conditions led us to the transcendental relation

$$\tan(z) = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$



$z_0 = 6.67$



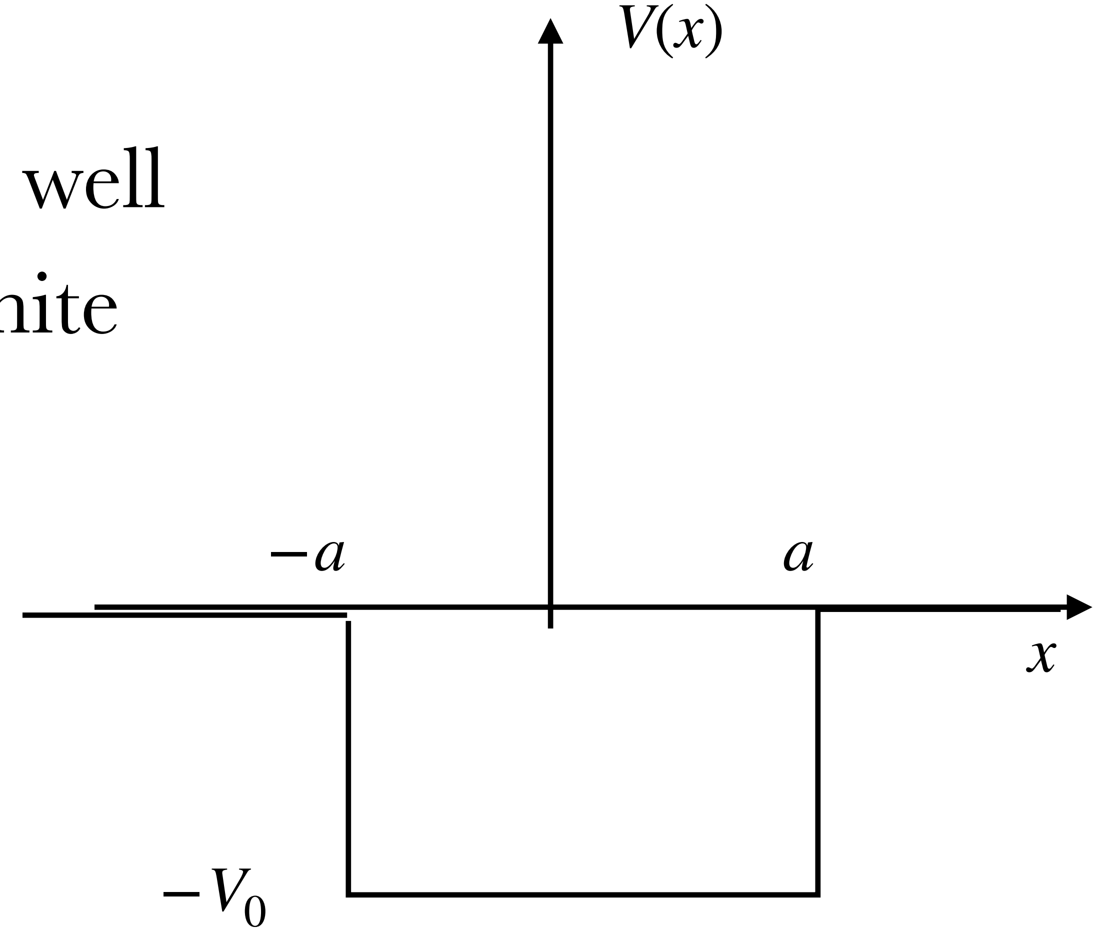
Intersections give the allowed z, which give allowed ℓ, which give allowed E.

Limiting cases: For a wide, deep well this should be related to the infinite square well.

$$z_{int} \approx z_n = \frac{n\pi}{2}$$

then, the allowed energies are:

$$z_n = \ell_n a = \frac{a\sqrt{2m(E_n + V_0)}}{\hbar} = \frac{n\pi}{2}$$



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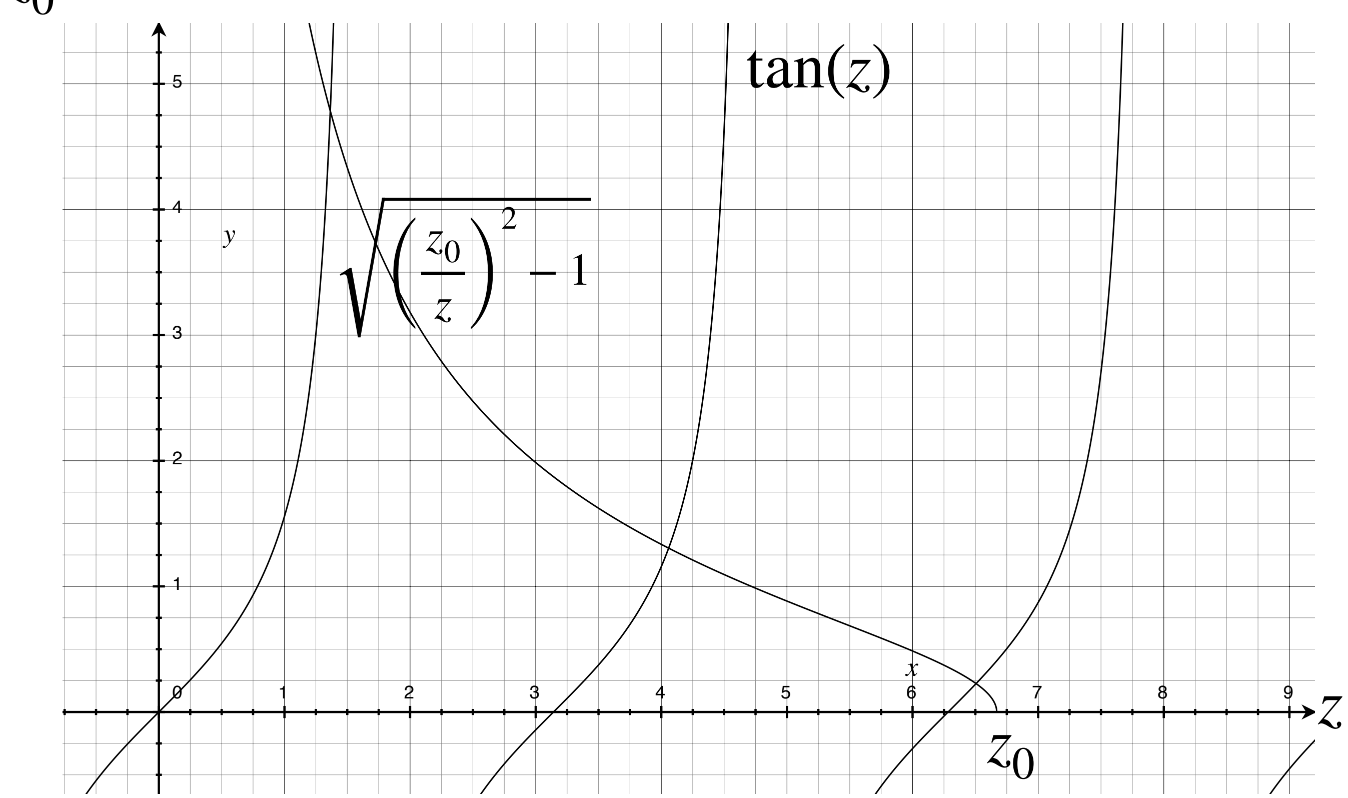


Or

$$E_n + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

These are the square well solutions above V_0, but only works for finitely many n.

In the narrow, shallow limit there's only one solution, but it's there!

Let's separate the final variables, x and y:

$\psi(x, y) = X(x)Y(y)$ separation ansatz

$$-\frac{\hbar^2}{2m}\left(Y(y)\frac{d^2X}{dx^2} + X(x)\frac{d^2Y}{dy^2}\right) + VXY = EXY$$

Dividing through by XY gives

$$-\frac{\hbar^2}{2m}\left(\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}\right) + V_x(x) + V_y(y) = E$$

This leads us to two separated equations

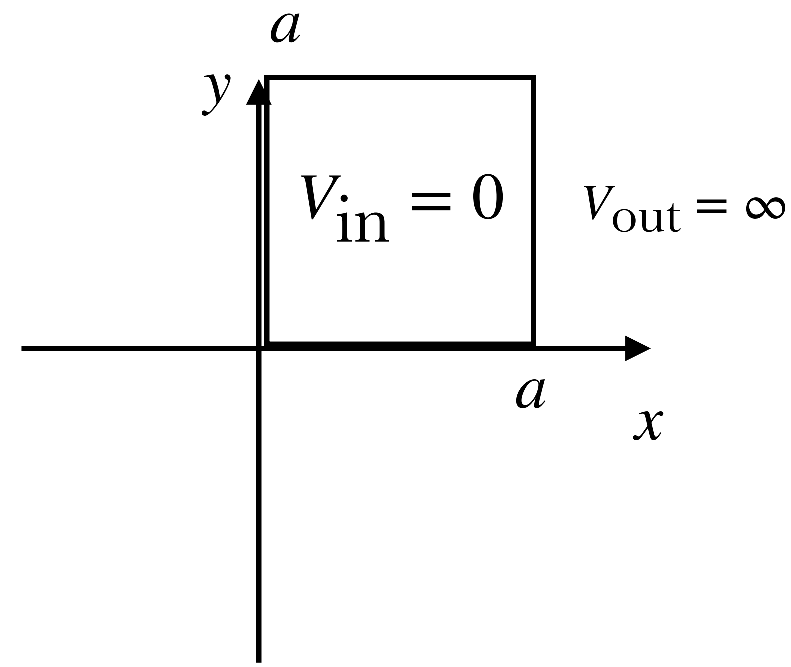
$$-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} + V_x(x) = \text{const.} = E_x$$

$$-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} + V_y(y) = \text{const.} = E_y$$

And $E_x + E_y = E$. These are just two copies of the 1D Sch. Eq.

Then the full solutions is $\psi(x, y) = X(x)Y(y)$.

II. 2D Infinite Square Well



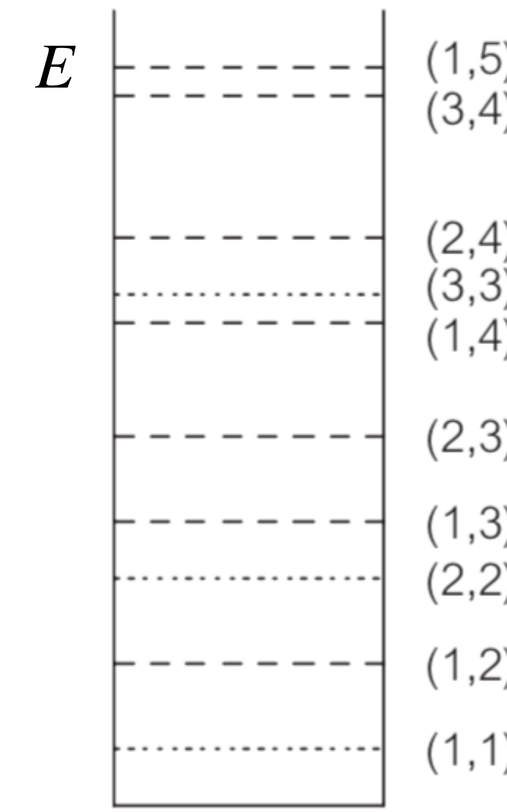
The potential:

$$V(x, y) = \begin{cases} 0 & \text{for } x, y \text{ between 0 and } a \\ \infty & \text{otherwise}. \end{cases}$$

Solutions for $X(x)$ and $Y(y)$:

$$X(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi}{a} x\right) \qquad E_x = \frac{n_x^2 \pi^2 \hbar^2}{2ma^2}$$

$$Y(y) = \sqrt{\frac{2}{a}} \sin\left(\frac{n_y \pi}{a} y\right) \qquad E_y = \frac{n_y^2 \pi^2 \hbar^2}{2ma^2}$$



Then $\psi(x, y)$ is

$$\psi(x, y) = X(x)Y(y) = \frac{2}{a} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right)$$

Then the total energy is

$$E = E_x + E_y = \frac{\pi^2 \hbar^2}{2ma^2}\left(n_x^2 + n_y^2\right) \qquad n_x, n_y = 1,2,3,\ldots.$$

Ground state has $(n_x, n_y) = (1,1)$. Notice also the two states (1,2) and (2,1) are degenerate (that is, they have the same E).

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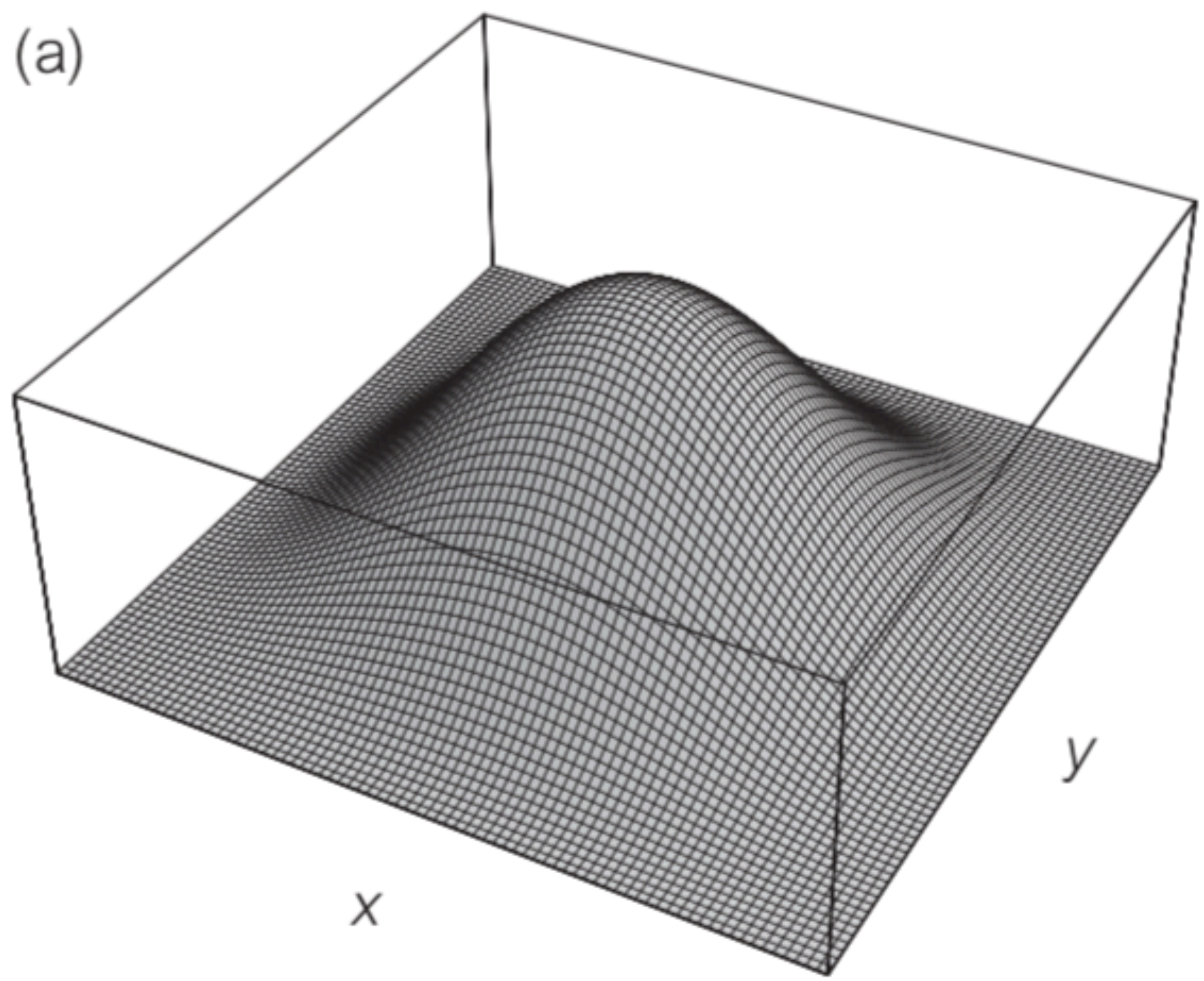
Let's sketch the probability density for the ground state, (1,1):

$\rho(x, y) = |\psi(x, y)|^2$



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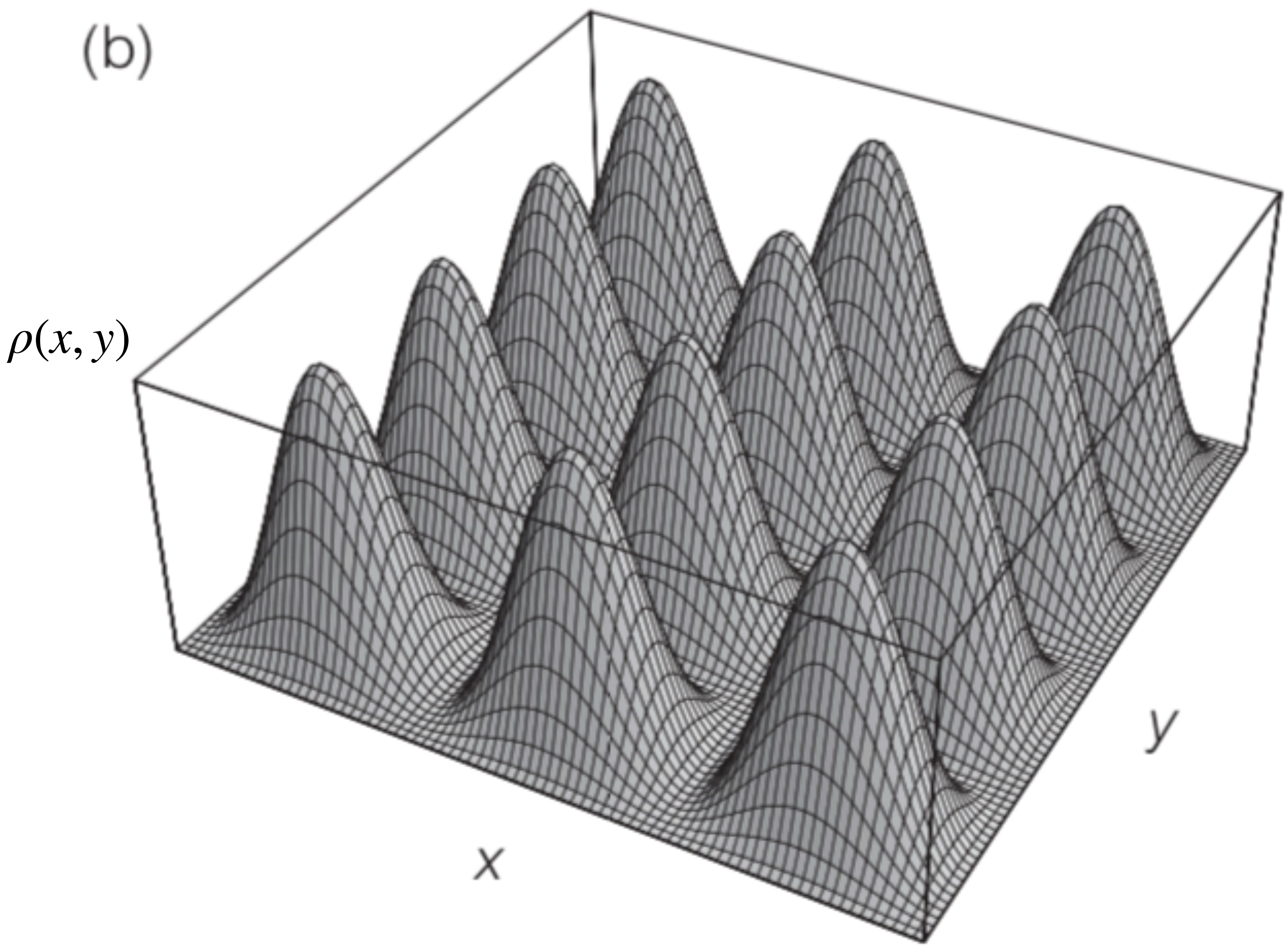
Let's sketch the probability density for the state, (3,4):



III. Angular Momentum

First classically: $\overrightarrow{L} = \vec{r} \times \overrightarrow{p} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$, let's focus on 2D and

hence on $L_z = xp_y - yp_x$.

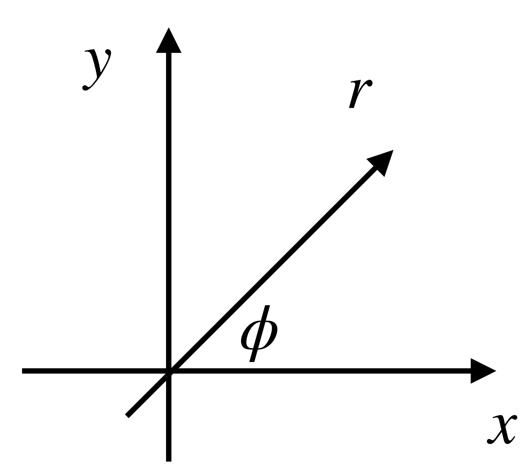
To go from the classical to the quantum theory, we introduce hats

$$L_z \longrightarrow \hat{L}_z \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

These expressions look even nicer in polar coordinates (r, ϕ)

Then $x = r\cos\phi$ and $y = r\sin\phi$

And $r = \sqrt{x^2 + y^2}$ and $\tan\phi = \dfrac{y}{x}$



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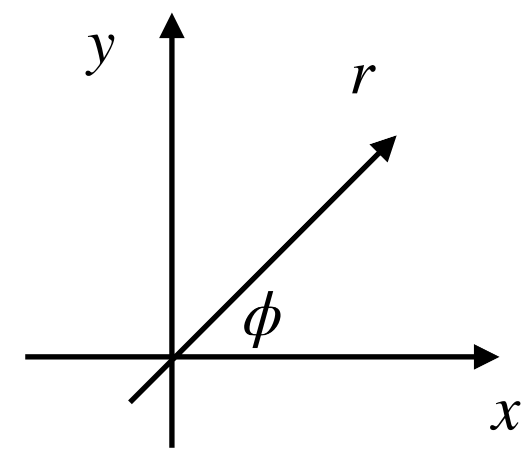

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}$$

Putting all of these into the definition of $\hat{L}_z$ gives

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$