Today

I. Last Time

II. Angular Momentum and the Circular Billiard

III. Algebraic Theory of Angular Momentum

I. Last time

* 2D Infinite square well

* Began studying angular momentum operators

$$L_z = (xp_y - yp_x) \longrightarrow \hat{L}_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

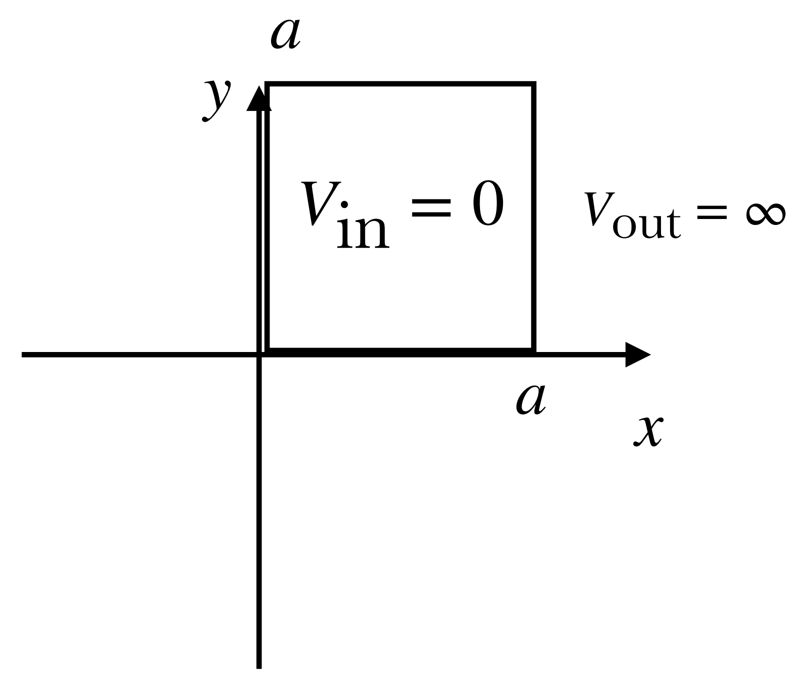
*Changed to polar coordinates

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

I. 2D Infinite Square Well

The potential:

$$V(x,y) = \begin{cases} 0 & \text{for } x, y \text{ between 0 and } a \\ \infty & \text{otherwise}. \end{cases}$$

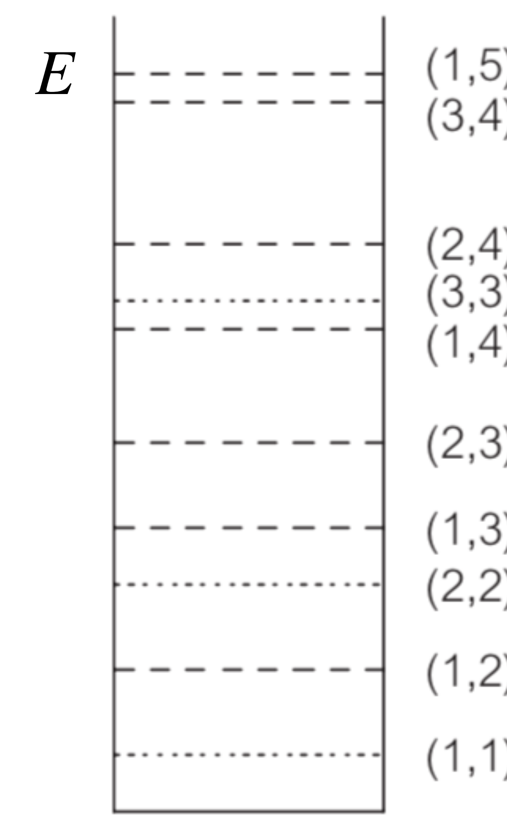


Wave functions were of a product form

$$\psi(x,y) = \frac{2}{a} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right)$$

The allowed energies were

$$E = \frac{\pi^2 \hbar^2}{2ma^2}\left(n_x^2 + n_y^2\right), \quad n_x, n_y = 1,2,3,\ldots.$$

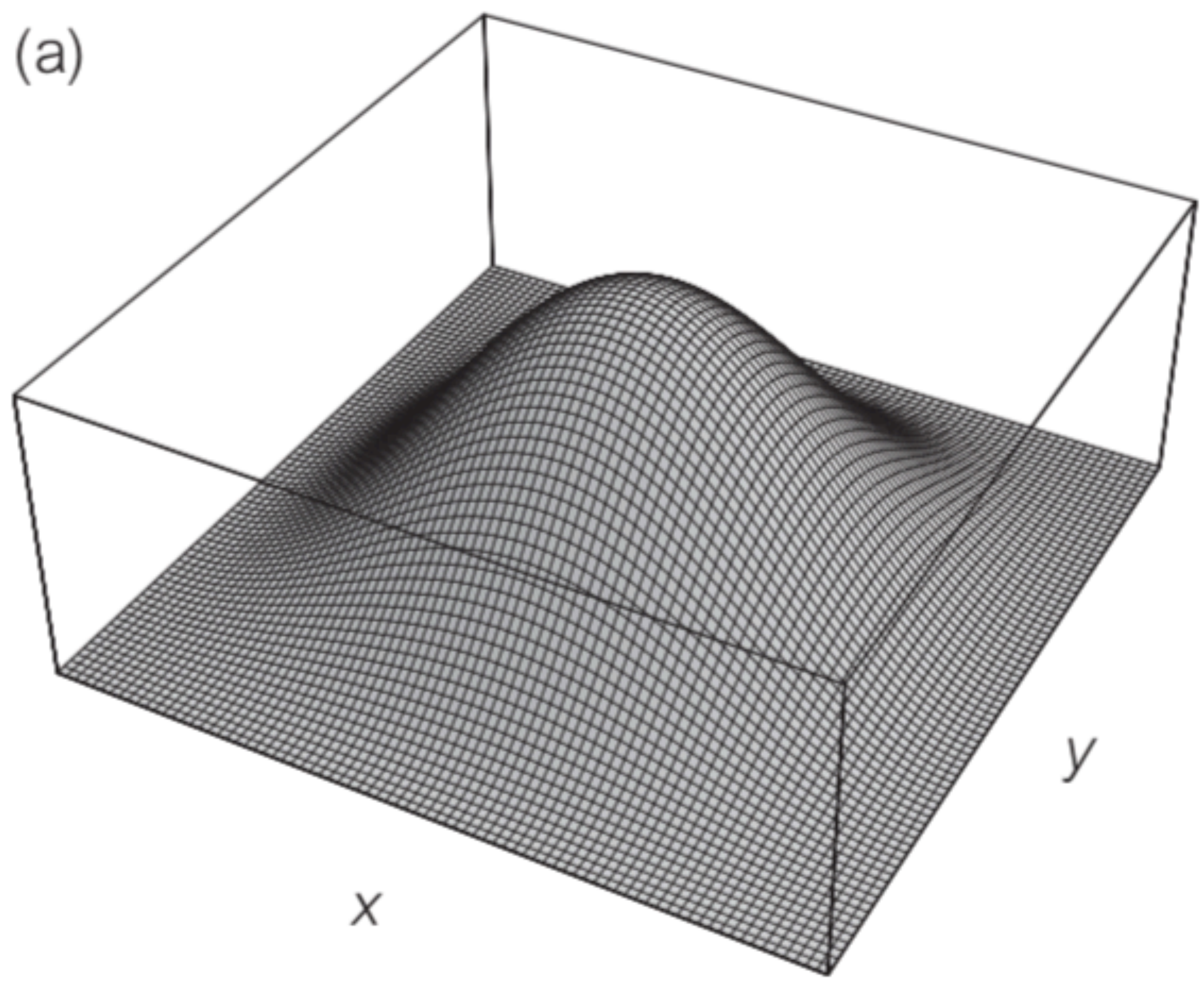


Then $\psi(x, y)$ is

$$\psi(x,y) = X(x)Y(y) = \frac{2}{a}\sin\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)$$

Let's sketch the probability density for the ground state, (1,1):

$\rho(x,y) = |\psi(x,y)|^2$



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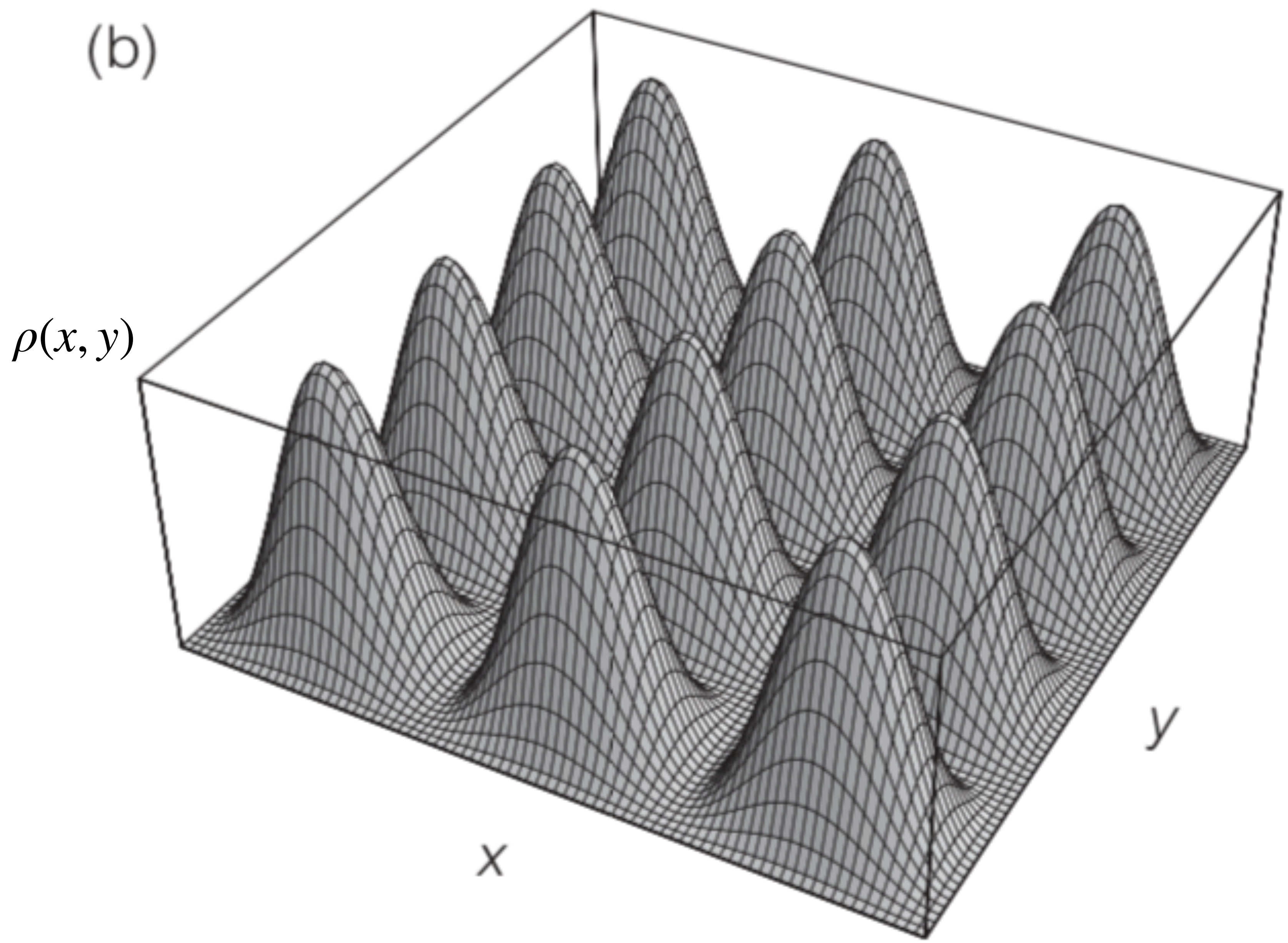
Let's sketch the probability density for the state, (3,4):



I. Angular Momentum

First classically: $\overrightarrow{L} = \vec{r} \times \overrightarrow{p} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$, let's focus on 2D and

hence on $L_z = xp_y - yp_x$.

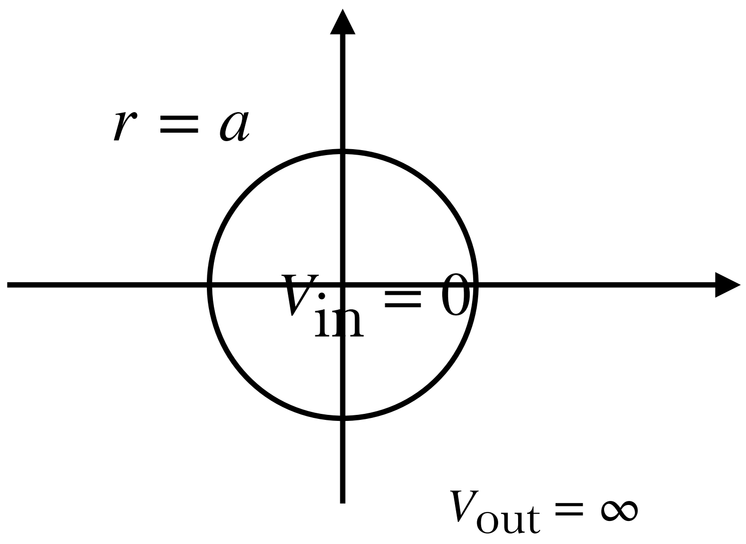
To go from the classical to the quantum theory, we introduce hats

$$L_z \longrightarrow \hat{L}_z \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Putting all of these into the definition of $\hat{L}_z$ gives

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

II. Angular momentum and the Circular Billiard



The potential:

$$V(r,\phi) = \begin{cases} 0 & \text{for } r \text{ between 0 and } a \text{ and for all } \phi \\ \infty & \text{otherwise}. \end{cases}$$

Note: Today we will call mass μ

The Schrodinger eqn. for this potential is :

$$\hat{H}\psi = E\psi$$

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 + V(r,\phi)$$

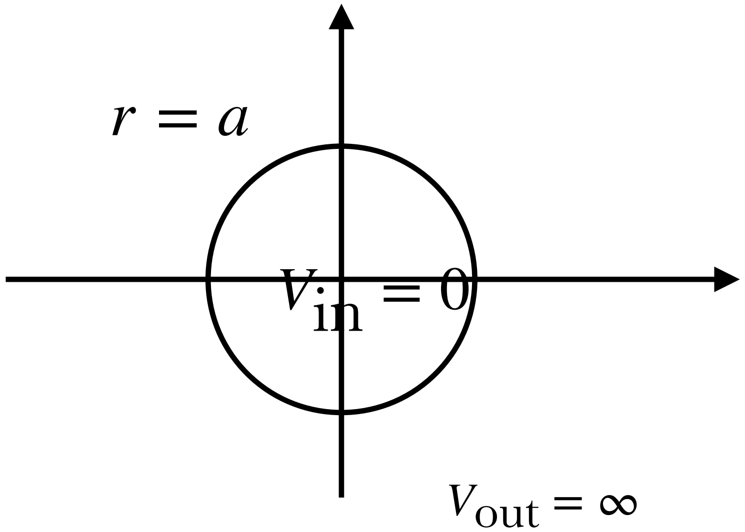
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

$$\left(-\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right] + V(r,\phi)\right)\psi = E\psi$$

To solve this we use separation of variables: $\psi(r,\phi) = R(r)\Phi(\phi)$

II. Angular momentum and the Circular Billiard

$$\left(-\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right]+V(r,\phi)\right)\psi = E\psi$$



To solve this we use separation of variables: $\psi(r,\phi) = R(r)\Phi(\phi)$

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi(\phi)$$

The radial solutions are described by Bessel functions!

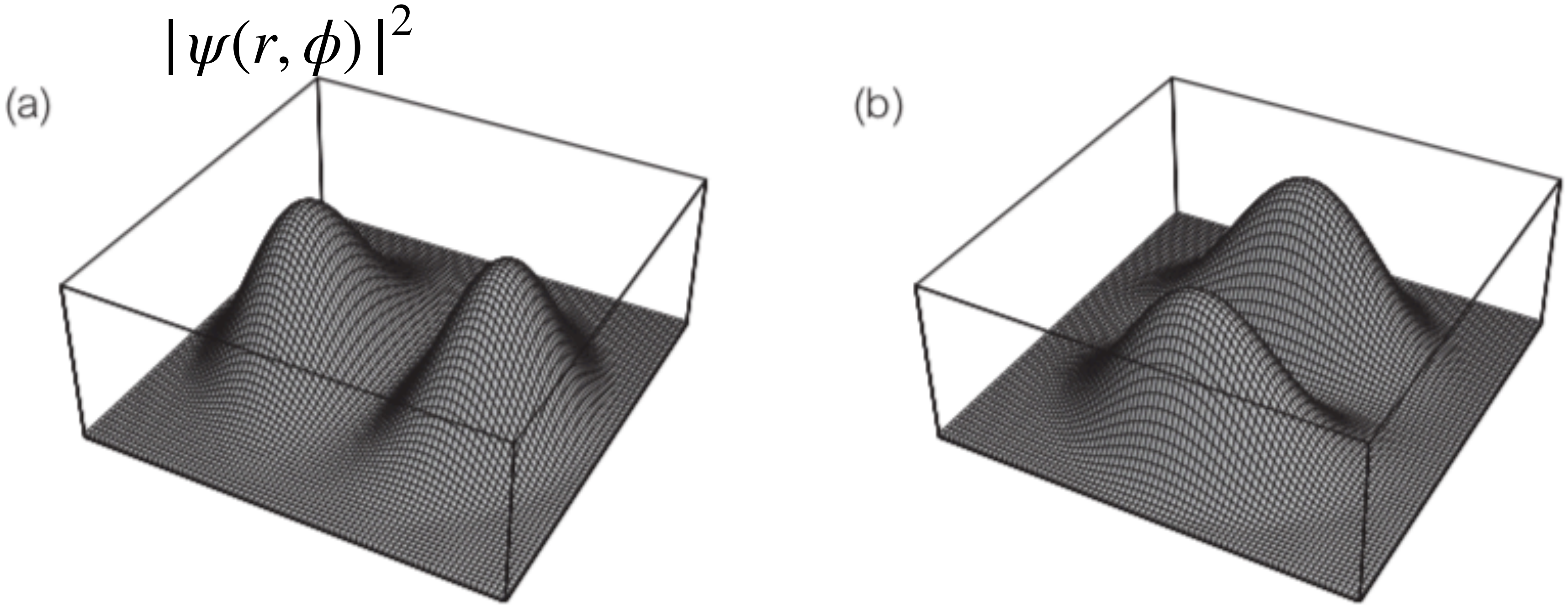
$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\left(V(r)+\frac{\hbar^2 m^2}{2\mu r^2}\right)R = ER$$

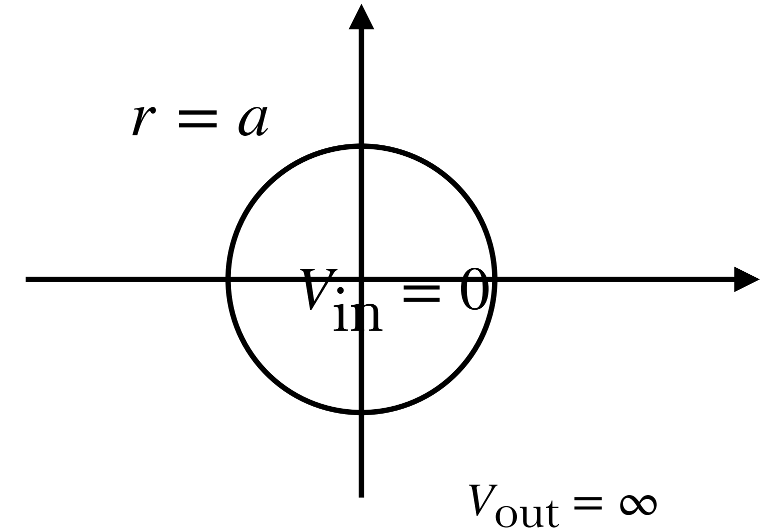
The solutions to the first equation are just:

$\Phi(\phi) = \frac{1}{\sqrt{2\pi}}e^{\pm im\phi}$, interesting because $\hat{L}_z\Phi = m\hbar\Phi$,

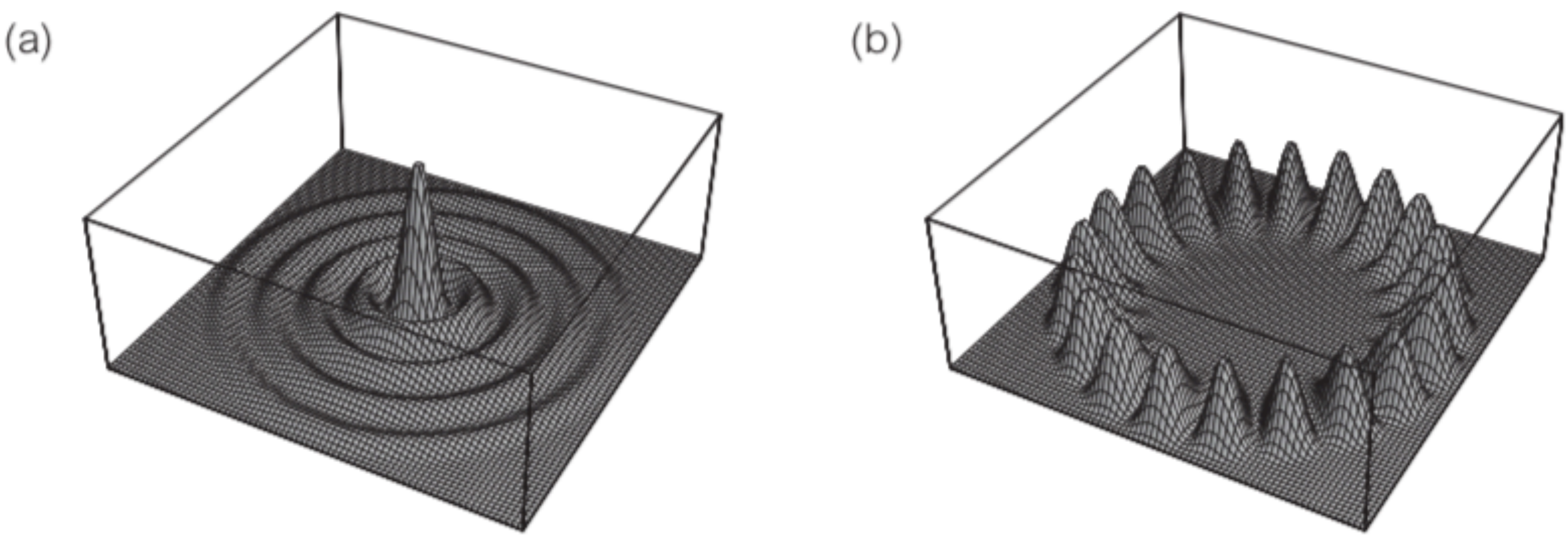
$m = \ldots, -2, -1, 0, 1, 2, \ldots.$

II. Angular momentum and the Circular Billiard





Two wave functions with $m = 1$



First has (n,m)=(4,0) and the second has (n,m)=(0,10)