Today

I. Last Time
II. Algebraic Theory of Angular Momentum
III. Angular Momentum Eigenvalues

I. Last time

* Studied the 2D circular billiard
* Separated variables in polar coordinates for the first time:

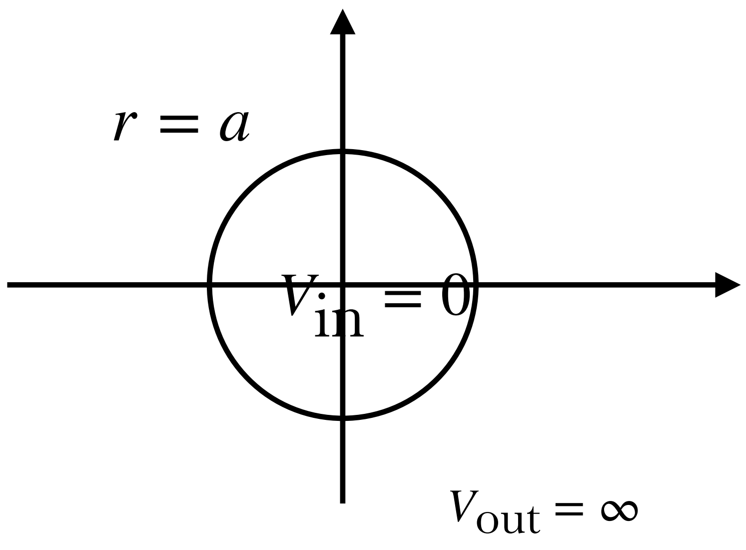
$$\psi(r,\phi) = R(r)\Phi(\phi)$$

* Found that the energy quantum number n described the number of radial rings
* The angular quantum number m, is called the 'magnetic' quantum number and describes the number of angular antinodes

I. Angular momentum and the Circular Billiard

The potential:

$$V(r,\phi) = \begin{cases} 0 & \text{for } r \text{ between 0 and } a \text{ and for all } \phi \\ \infty & \text{otherwise}. \end{cases}$$



Note: Today we will call mass μ

The Schrodinger eqn. for this potential is :

$$\hat{H}\psi = E\psi$$

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 + V(r,\phi)$$

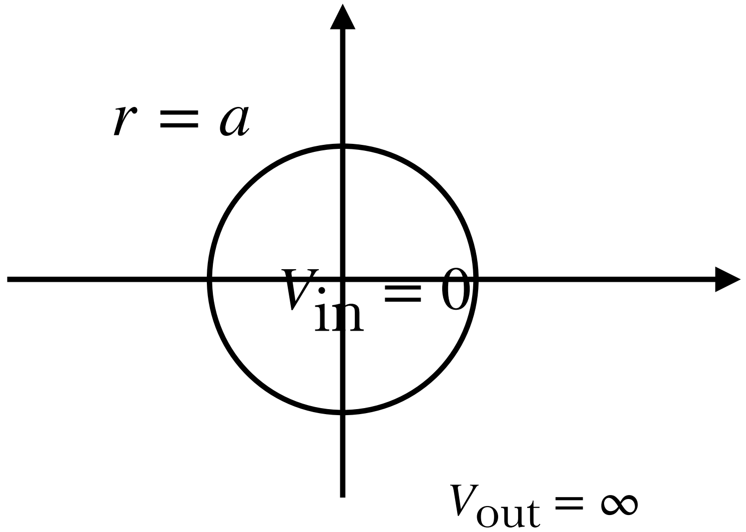
$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

$$\left(-\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right] + V(r,\phi)\right)\psi = E\psi$$

To solve this we use separation of variables: $\psi(r,\phi) = R(r)\Phi(\phi)$

I. Angular momentum and the Circular Billiard

$$\left(-\frac{\hbar^2}{2\mu}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right]+V(r,\phi)\right)\psi = E\psi$$



To solve this we use separation of variables: $\psi(r,\phi) = R(r)\Phi(\phi)$

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi(\phi)$$

The radial solutions are described by Bessel functions!

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right)+\left(V(r)+\frac{\hbar^2 m^2}{2\mu r^2}\right)R = E_n R$$

Abramowitz and Stegun.

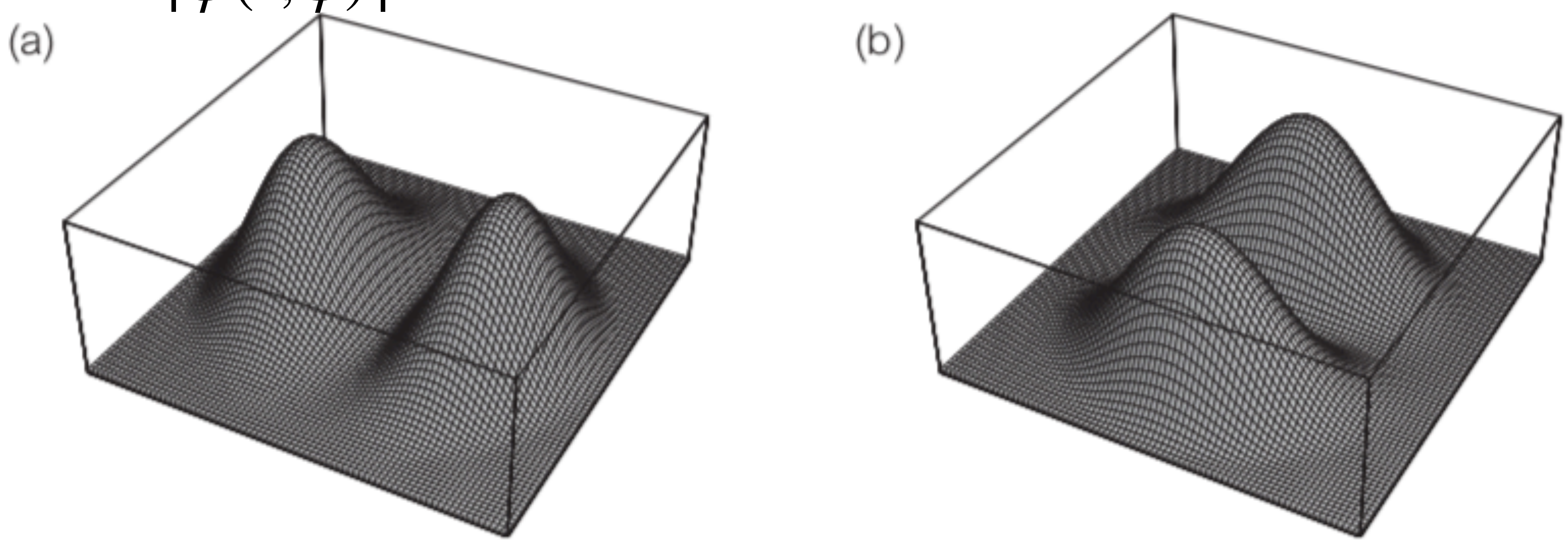
The solutions to the first equation are just:

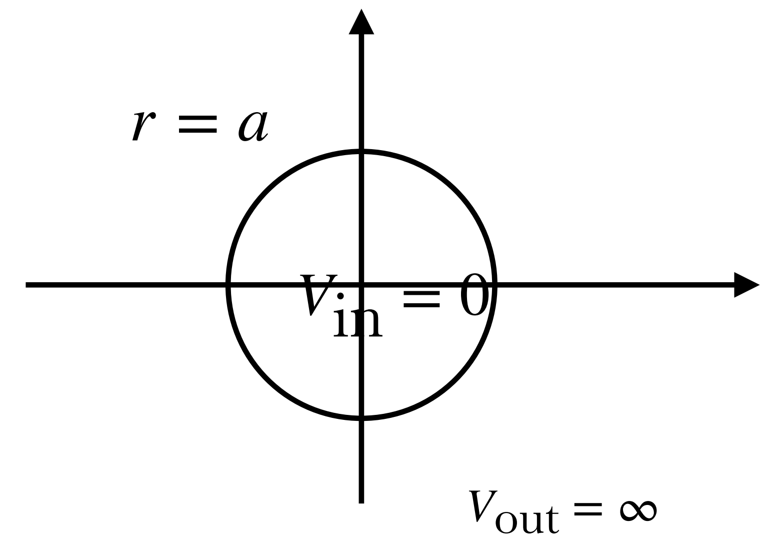
$\Phi(\phi) = \frac{1}{\sqrt{2\pi}}e^{\pm im\phi}$, interesting because $\hat{L}_z\Phi = m\hbar\Phi$,

$m = \ldots, -2, -1, 0, 1, 2, \ldots$.

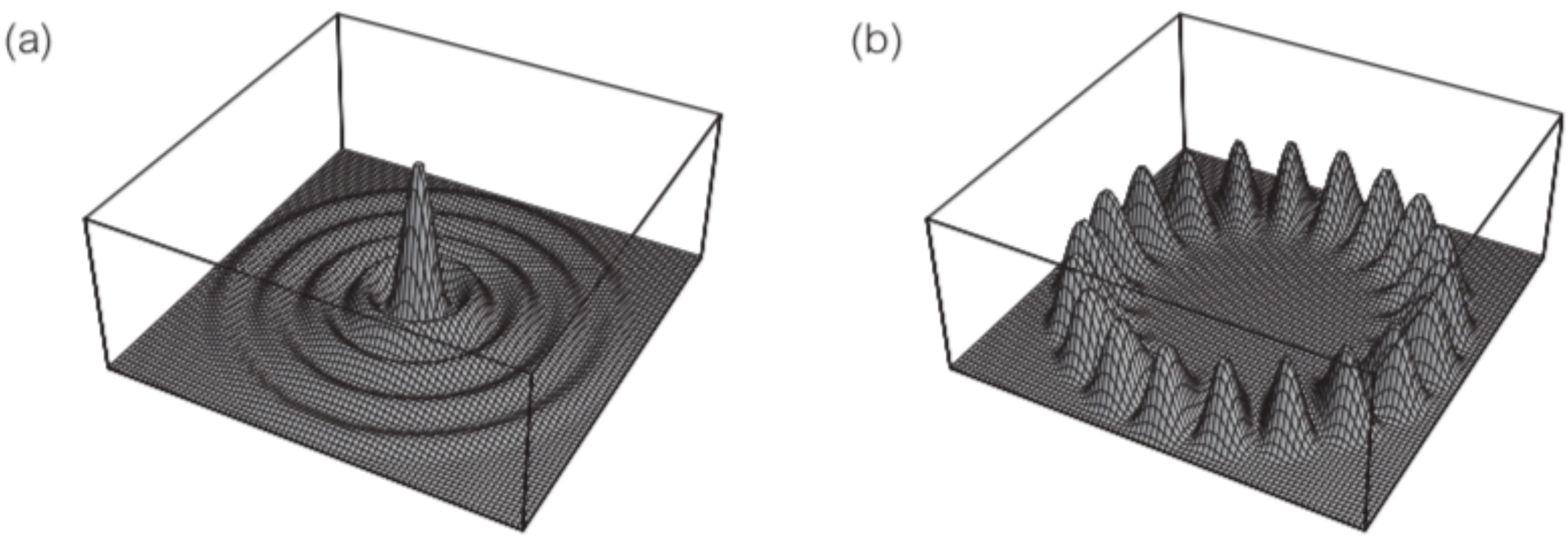
I. Angular momentum and the Circular Billiard

$|\psi(r, \phi)|^2$





Two wave functions with $m = 1$



First has (n,m)=(4,0) and the second has (n,m)=(0,10)

I. Angular Momentum

First classically: $\overrightarrow{L} = \vec{r} \times \overrightarrow{p} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$, let's focus on 2D and

hence on $L_z = xp_y - yp_x$.

To go from the classical to the quantum theory, we introduce hats
$$L_z \longrightarrow \hat{L}_z \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Putting all of these into the definition of $\hat{L}_z$ gives
$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

II. Algebraic Theory of Angular Momentum

First classically: $\vec{L} = \vec{r} \times \vec{p} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$, let's focus on 3D

today: $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, $L_z = xp_y - yp_x$.

To go from the classical to the quantum theory, we introduce hats

$$L_x \longrightarrow \hat{L}_x \equiv \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y \longrightarrow \hat{L}_y \equiv \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z \longrightarrow \hat{L}_z \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

For the rest of today, I'll drop hats. You'll remember they're operators.

$$L_x \longrightarrow \hat{L}_x \equiv \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y \longrightarrow \hat{L}_y \equiv \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z \longrightarrow \hat{L}_z \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Do these operators commute?

$$[A, B + C] = [A, B] + [A, C] \qquad [A, BC] = [A, B]C + B[A, C]$$

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x - xp_z] - [zp_y, zp_x - xp_z]$$

$$= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]$$

$$= y[p_z, zp_x] + x[z, p_z]p_y$$

$$= y[p_z, z]p_x + yz[p_z, p_x] + x[z, p_z]p_y$$

$$= y[p_z, z]p_x + x[z, p_z]p_y = -y[z, p_z]p_x + x[z, p_z]p_y = i\hbar(-yp_x + xp_y)$$

$$= i\hbar L_z$$

Conclusion(!): $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$ and cyclic permutations.