Today

I. Last Time

II. Algebraic Theory of Angular Momentum Continued

III. Angular Momentum Eigenvalues

I. Last time

* Intro'd all three angular momentum operators
* Found $[L_x, L_y] = i\hbar L_z$
* This means that we cannot simultaneously measure L_x and L_y.
* $[A, BC] = B[A, C] + [A, B]C$
* Notation: I've begun to drop hats everywhere.

I. Algebraic Theory of Angular Momentum

First classically: $\overrightarrow{L} = \vec{r} \times \overrightarrow{p} = \det \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$, let's focus on 3D

today: $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, $L_z = xp_y - yp_x$.

To go from the classical to the quantum theory, we introduce hats

$$L_x \longrightarrow \hat{L}_x \equiv \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y \longrightarrow \hat{L}_y \equiv \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z \longrightarrow \hat{L}_z \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

For the rest of today, I'll drop hats. You'll remember they're operators.

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Do these operators commute?

$$[A, B + C] = [A, B] + [A, C] \qquad [A, BC] = [A, B]C + B[A, C]$$

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z] = [yp_z, zp_x - xp_z] - [zp_y, zp_x - xp_z]$$

$$= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]$$

$$= y[p_z, zp_x] + x[z, p_z]p_y$$

$$= y[p_z, z]p_x + yz[p_z, p_x] + x[z, p_z]p_y$$

$$= y[p_z, z]p_x + x[z, p_z]p_y = -y[z, p_z]p_x + x[z, p_z]p_y = i\hbar(-yp_x + xp_y)$$

$$= i\hbar L_z$$

Conclusion(!): $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$ and cyclic permutations.

II. Conclusion(!): $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$ and cyclic permutations.

If two operators don't commute, then there's always an uncertainty relation for them:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2 .$$

Let's apply this to the case of angular momentum:

$$\sigma_{L_x} \sigma_{L_y} \geq \left| \left(\frac{1}{2i} \langle i\hbar L_z \rangle \right) \right| = \frac{\hbar}{2} | \langle L_z \rangle | .$$

There's a way out of this complicated situation (really only a partial way out). We want to try and find some quantity that does commute with the components of $\overrightarrow{L}$. The idea is to turn to the magnitude of $\overrightarrow{L}$, $L^2 = \overrightarrow{L} \cdot \overrightarrow{L} = L_x^2 + L_y^2 + L_z^2$. Let's compute

$$[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] + [L_z^2, L_z]$$

$$
\begin{aligned}
[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z] &= [L_x^2, L_z] + [L_y^2, L_z] + [L_z^2, L_z] \\
&= [L_x^2, L_z] + [L_y^2, L_z] \\
&= L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y \\
&= -i\hbar L_x L_y - i\hbar L_y L_x + L_y(i\hbar L_x) + i\hbar L_x L_y \\
&= 0. \checkmark
\end{aligned}
$$

Then, because none of the directions is special, we expect

$$[L^2, \overrightarrow{L}] = 0!$$

So, L_z and L^2 are compatible observables. Let's find their eigenvalues.

III. Let's call the eigenvalues μ and λ respectively. More concretely

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We'll try to copy what we did for the oscillator:

$$L_\pm \equiv L_x \pm i L_y.$$

Then,

$$\begin{aligned} [L_z, L_\pm] &= [L_z, L_x] \pm i[L_z, L_y] \\ &= i\hbar L_y \pm i(-i\hbar L_x) \end{aligned}.$$

So,

$$[L_z, L_\pm] = \pm\, \hbar(L_x \pm i L_y) = \pm\, \hbar L_\pm.$$

$$[L^2, L_\pm] = 0.$$

<u>Claim</u>: If f satisfies Eq. (1), then so does $L_\pm f$, except with

$L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)$. <u>Pf:</u>

$L^2(L_\pm f) = L_\pm(L^2 f) = L_\pm(\lambda f) = \lambda(L_\pm f)\,.$ ✓

$L_z(L_\pm f) = (L_z L_\pm - L_\pm L_z) f + L_\pm L_z f = \pm\, \hbar L_\pm f + \mu L_\pm f = (\mu \pm \hbar) L_\pm\,.$ ✓