Today

I. Last Time

II. Angular Momentum Eigenvalues

III. Begin Derivation of the Angular Momentum Eigenfunctions

I. Last time

* Looked at commutativity of angular momentum components: $[L_x, L_y] = i\hbar L_z$ and cyclic permutations

* We also found: $[\overrightarrow{L}, L^2] = 0$. This means that we can simultaneously observe any one component of $\overrightarrow{L}$ and L^2.

* Introduced ladder operators for angular moment:

$L_\pm = L_x \pm iL_y$ and found their commutation properties

$[L_\pm, L^2] = 0$, and $[L_z, L_\pm] = \pm\, \hbar L_\pm$.

* Two partial results on eigenvalues: $L^2(L_\pm f) = \lambda(L_\pm f)$,

$L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)$. These operators raise and lower the z-component of angular momentum.

I. Conclusion(!): $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$ and cyclic permutations.

If two operators don't commute, then there's always an uncertainty relation for them:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2.$$

Let's apply this to the case of angular momentum:

$$\sigma_{L_x} \sigma_{L_y} \geq \left| \left(\frac{1}{2i} \langle i\hbar L_z \rangle \right) \right| = \frac{\hbar}{2} | \langle L_z \rangle | .$$

There's a way out of this complicated situation (really only a partial way out). We want to try and find some quantity that does commute with the components of $\vec{L}$. The idea is to turn to the magnitude of $\vec{L}$, $L^2 = \vec{L} \cdot \vec{L} = L_x^2 + L_y^2 + L_z^2$. Let's compute

$$[L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] + [L_z^2, L_z]$$

$$\text{I. } [L^2, L_z] = [L_x^2 + L_y^2 + L_z^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] + [L_z^2, L_z]$$
$$= [L_x^2, L_z] + [L_y^2, L_z]$$
$$= L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y$$
$$= -i\hbar L_x L_y - i\hbar L_y L_x + L_y(i\hbar L_x) + i\hbar L_x L_y$$
$$= 0. \checkmark$$

Then, because none of the directions is special, we expect

$$[L^2, \overrightarrow{L}] = 0!$$

So, L_z and L^2 are compatible observables. Let's find their eigenvalues.

Let's call the eigenvalues μ and λ respectively. More concretely

$$L_z f = \mu f \quad \text{and} \quad L^2 f = \lambda f. \qquad (1)$$

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We'll try to copy what we did for the oscillator:

$$L_\pm \equiv L_x \pm i L_y.$$

Then,

$$\begin{aligned}[L_z, L_\pm] &= [L_z, L_x] \pm i[L_z, L_y] \\ &= i\hbar L_y \pm i(-i\hbar L_x)\end{aligned} \; .$$

So,

$$[L_z, L_\pm] = \pm\,\hbar(L_x \pm iL_y) = \pm\,\hbar L_\pm.$$

$$[L^2, L_\pm] = 0.$$

<u>Claim</u>: If f satisfies Eq. (1), then so does $L_\pm f$, except with

$L_z(L_\pm f) = (\mu \pm \hbar)(L_\pm f)$. <u>Pf:</u>

$L^2(L_\pm f) = L_\pm(L^2 f) = L_\pm(\lambda f) = \lambda(L_\pm f)\,.$ ✓

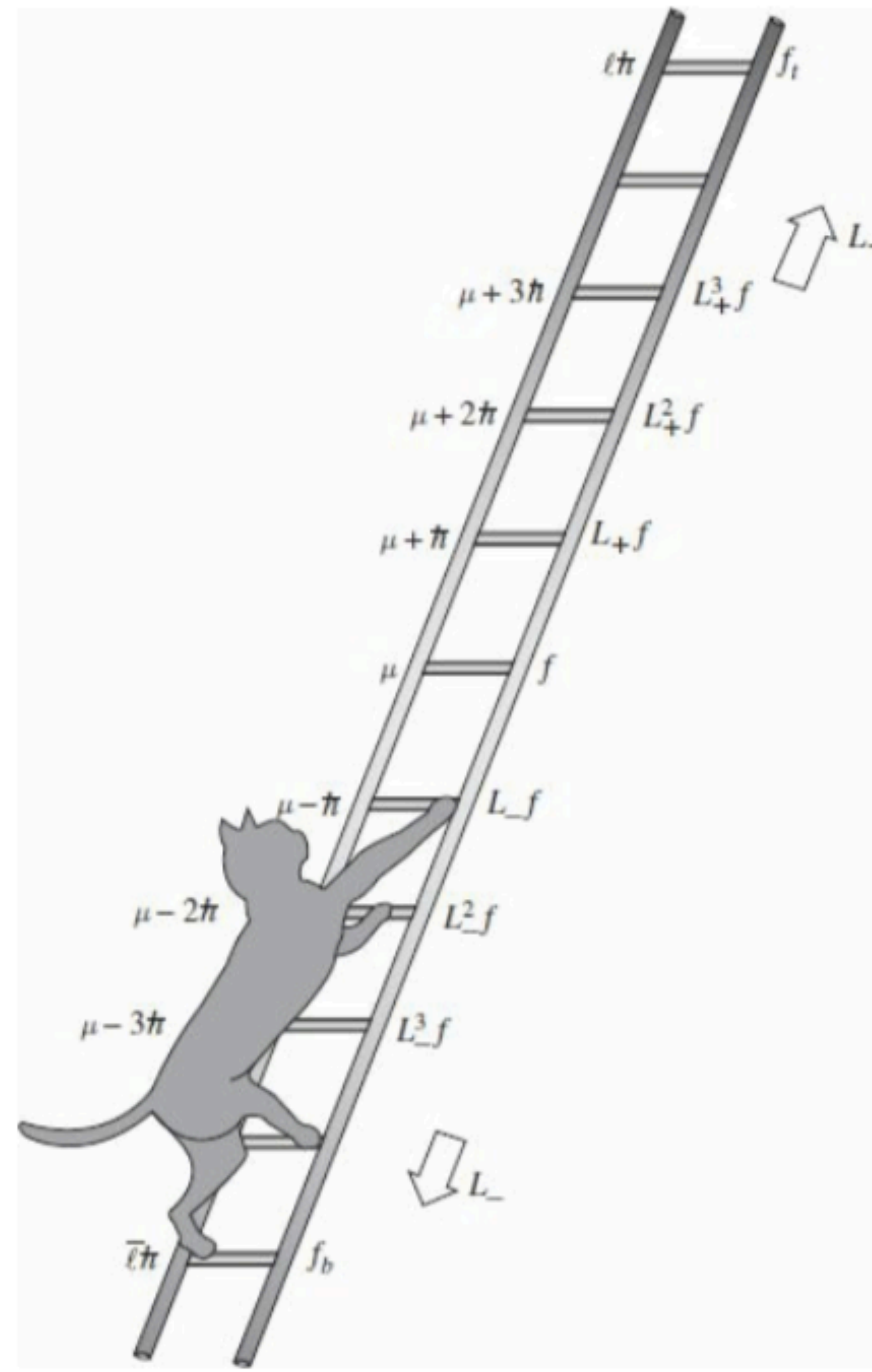
$L_z(L_\pm f) = (L_z L_\pm - L_\pm L_z) f + L_\pm L_z f = \pm\,\hbar L_\pm f + \mu L_\pm f = (\mu \pm \hbar) L_\pm\,.$ ✓

II. Let's use the ladder operators to find the angular momentum eigenvalues.

Notice that we can't raise L_z indefinitely. So, eventually we hit a top rung:

$$L_+ f_t = 0.$$

Let's give a name to the z-component of angular momentum of the top rung, $L_z f_t = \ell\hbar f_t$ and $L^2 f_t = \lambda f_t$. To relate ℓ and λ let's try to find a relation between L^2, $L_\pm$, and L_z. Let's compute



$$L_\pm L_\mp = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x)$$

$$= L_x^2 + L_y^2 \mp i(i\hbar L_z)$$

$$= L^2 - L_z^2 \pm \hbar L_z.$$

Solve this for L^2, to get

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z.$$

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$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z.$

It follows that

$$L^2 f_t = (L_- L_+ + L_z^2 + \hbar L_z) f_t = (0 + \ell^2\hbar^2 + \ell\hbar^2) f_t = \hbar^2 \ell(\ell+1) f_t$$

So, $\lambda = \hbar^2 \ell(\ell + 1)$

Similarly, $L_- f_b = 0$ and

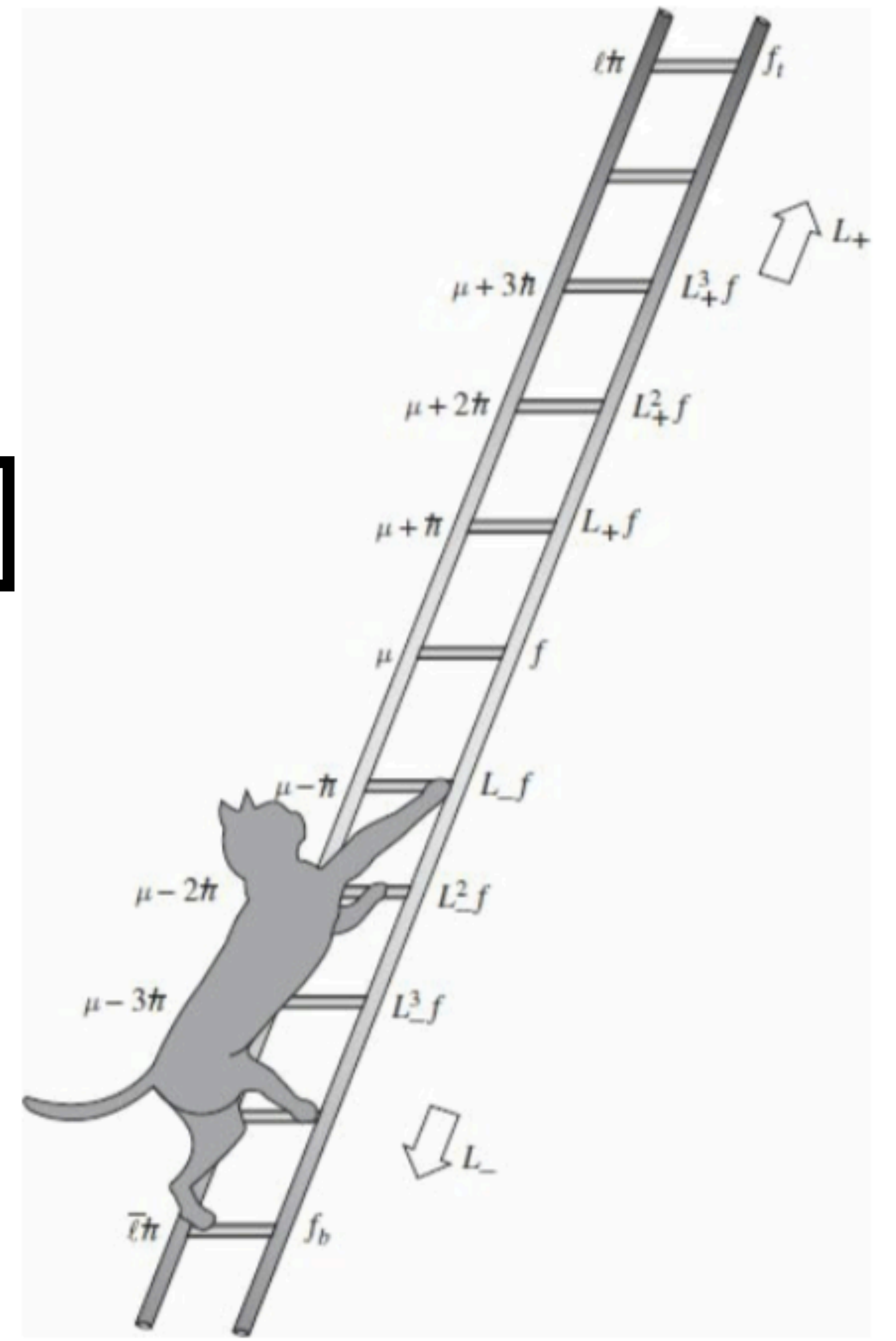
$$L^2 f_b = (L_+ L_- + L_z^2 - \hbar L_z) f_b = (0 + \bar{\ell}^2\hbar^2 - \bar{\ell}\hbar^2) f_b = \hbar^2 \bar{\ell}(\bar{\ell} - 1) f_b$$

Then, we can also write,

$\lambda = \hbar^2 \bar{\ell}(\bar{\ell} - 1) \implies \bar{\ell}(\bar{\ell} - 1) = \ell(\ell + 1).$

So,

$$\bar{\ell} = \begin{cases} \ell + 1 & \text{physically absurd} \\ -\ell & \text{yes!} \end{cases}$$



So,

$$\bar{\ell} = \begin{cases} \ell + 1 & \text{physically absurd} \\ -\ell & \text{yes!} \end{cases}$$

We conclude that the eigenvalues of L_z are $m\hbar$ where

$$m = -\ell, -\ell + 1, -\ell + 2, \ldots, \ell - 1, \ell$$

goes in integer steps. We must have that $\ell = -\ell + N$, where N is an integer. So,

$\ell = \frac{N}{2}$ is either an integer or a half-integer.

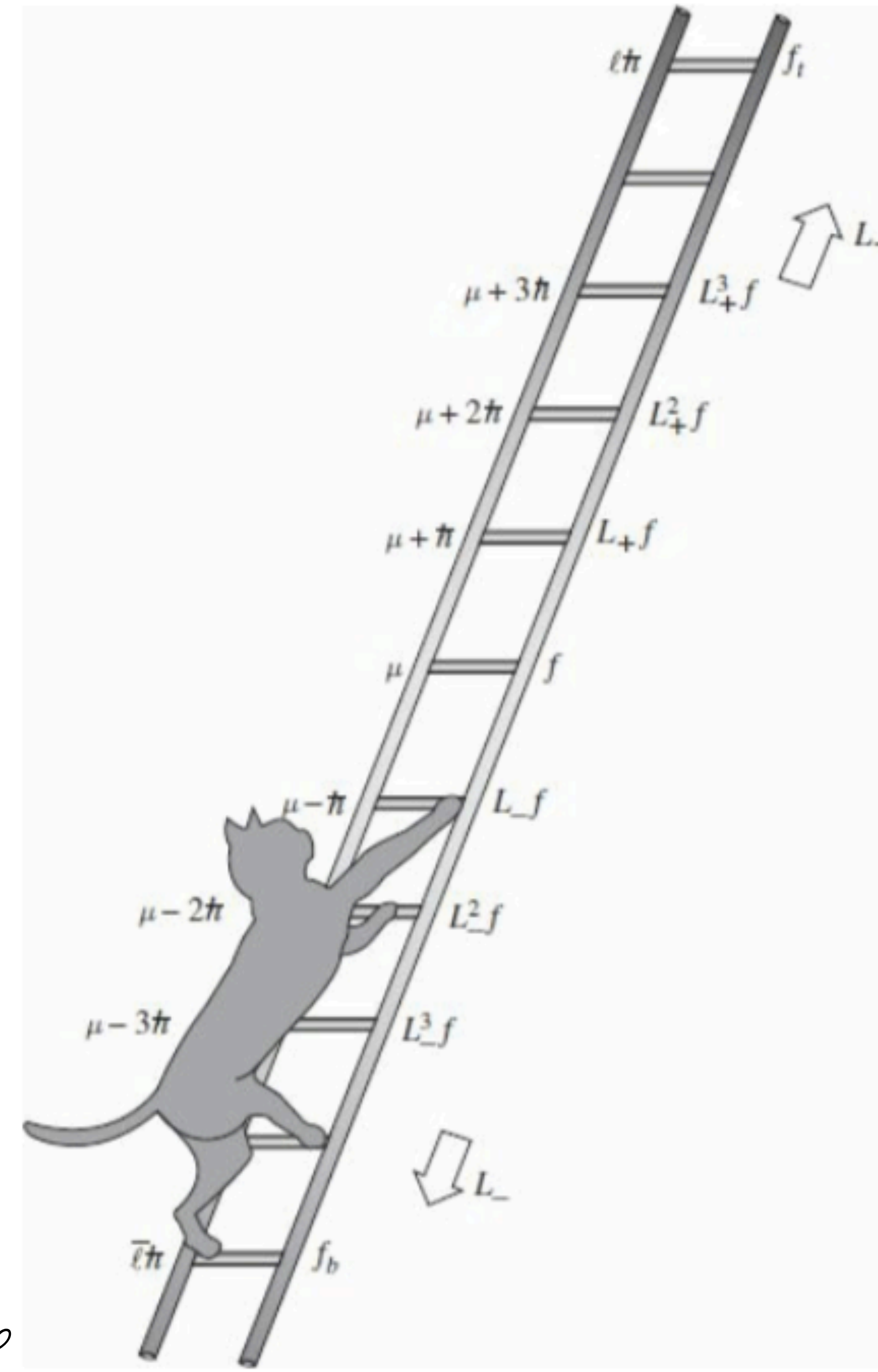
Intro notation

$$L^2 f_\ell^m = \hbar^2 \ell(\ell + 1) f_\ell^m$$

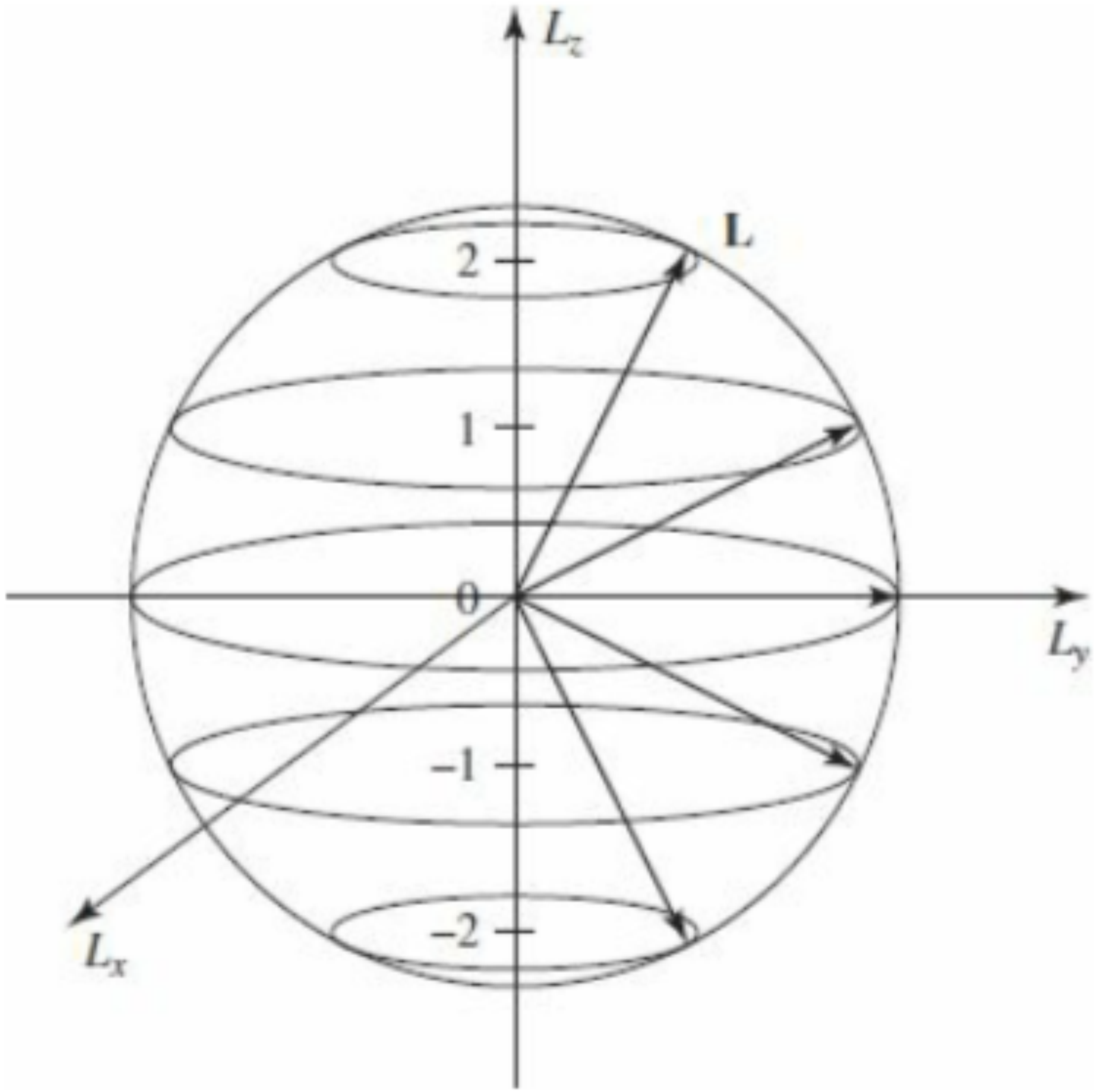
and

$L_z f_\ell^m = \hbar m f_\ell^m$, where

$$\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots; \; m = -\ell, -\ell + 1, \ldots, \ell - 1, \ell$$



We can roughly picture this as follows:



The radius of this sphere $\sqrt{\ell(\ell+1)}$ is in general greater than $\max(L_z)$.