Today

I. Last Time

II. Derivation of the Angular Momentum Eigenfunctions

III. Radial Equation

I. Last time

* We found the ladder operators for angular momentum:

$L_{\pm} = L_x \pm iL_y$

*Found angular momentum eigenvalues

$L_z f_{\ell}^{m} = m\hbar f_{\ell}^{m}, \quad m = -\ell, -\ell+1, \cdots, \ell-1, \ell,$ where ℓ is an integer or half integer.

$L^2 f_{\ell}^{m} = \ell(\ell+1) f_{\ell}^{m}, \quad \ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots.$

*$[L^2, \overrightarrow{L}] = 0$, $[L_x, L_y] = i\hbar L_z$ and cyclic, $[L_z, L_{\pm}] = \pm \hbar L_{\pm}$.

$L^2 = L_{\pm}L_{\mp} + L_z^2 \mp \hbar L_z.$

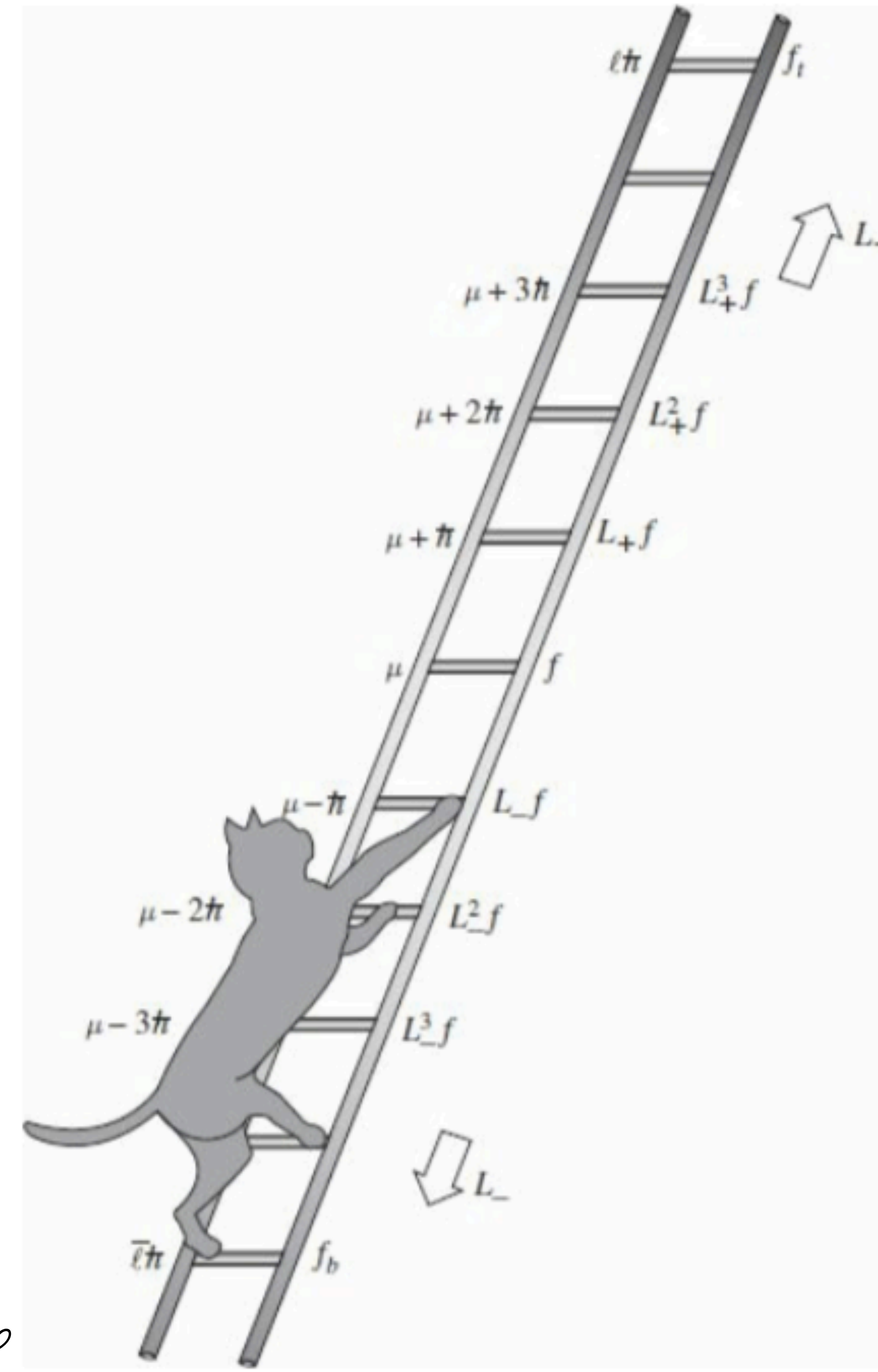
So,

$$\bar{\ell} = \begin{cases} \ell + 1 & \text{physically absurd} \\ -\ell & \text{yes!} \end{cases}$$

We conclude that the eigenvalues of L_z are $m\hbar$ where

$$m = -\ell, -\ell + 1, -\ell + 2, \ldots, \ell - 1, \ell$$

goes in integer steps. We must have that $\ell = -\ell + N$, where N is an integer. So, $\ell = \dfrac{N}{2}$ is either an integer or a half-integer.



Intro notation

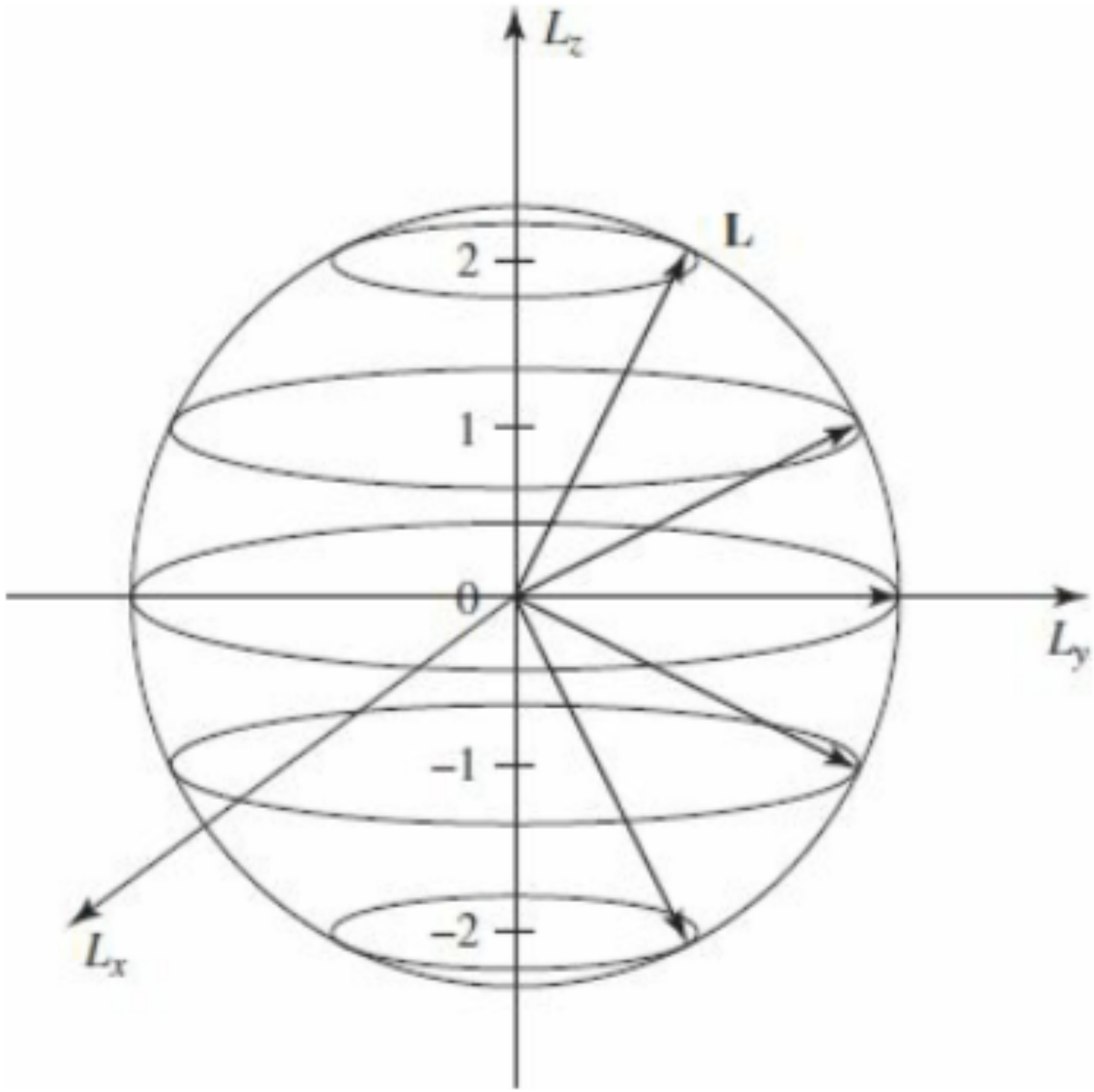
$$L^2 f_\ell^m = \hbar^2 \ell(\ell + 1) f_\ell^m$$

and

$$L_z f_\ell^m = \hbar m f_\ell^m,$$ where

$$\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots; \; m = -\ell, -\ell + 1, \ldots, \ell - 1, \ell$$

We can roughly picture this as follows:



The radius of this sphere $\sqrt{\ell(\ell+1)}$ is in general greater than $\max(L_z)$.

II. Derivation of angular momentum eigenfunctions.

Classically, $\overrightarrow{L} = \vec{r} \times \overrightarrow{p}$. We want to convert this into operators

$$\hat{\overrightarrow{L}} = \frac{\hbar}{i} \vec{r} \times \overrightarrow{\nabla}.$$

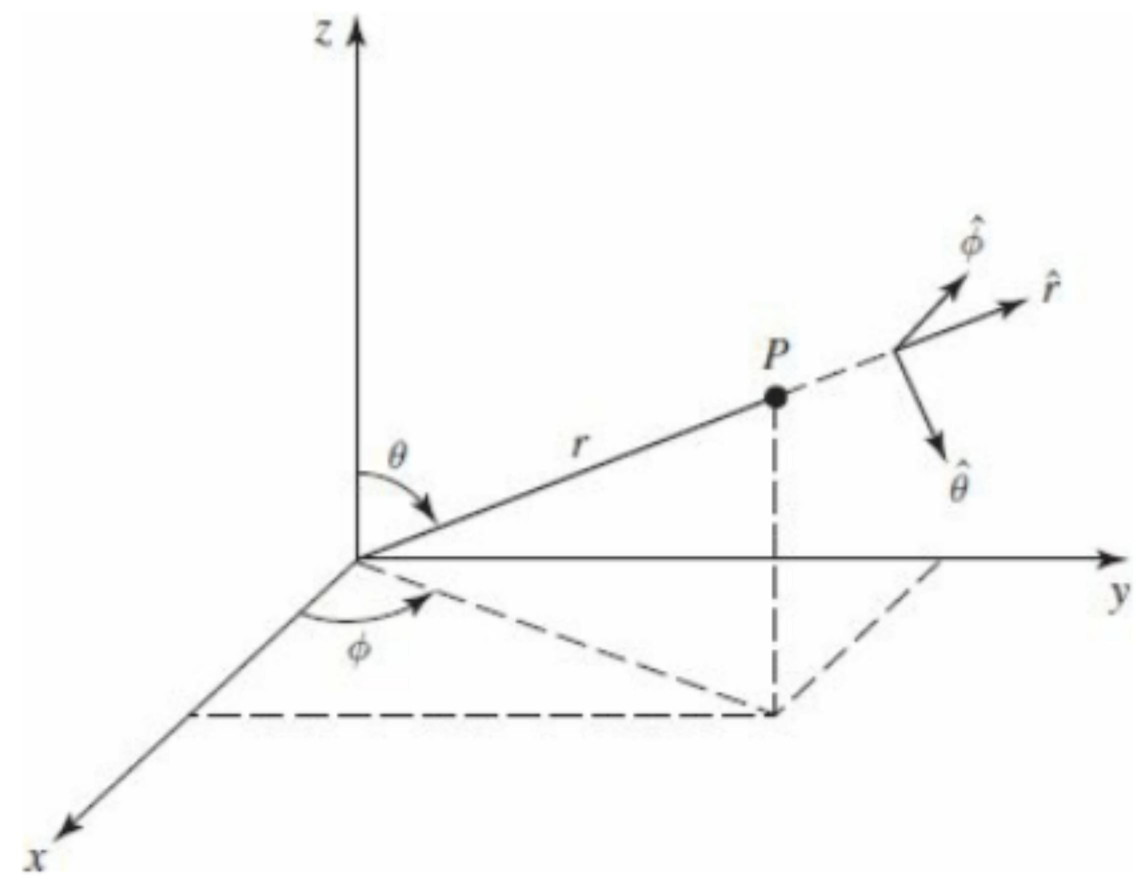
In spherical coords. (natural for ang. momentum and rotations)

$$\overrightarrow{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}.$$

Now, $\vec{r} = r\hat{r}$, so

$$\overrightarrow{L} = \frac{\hbar}{i}\left[r(\hat{r}\times\hat{r})\frac{\partial}{\partial r} + (\hat{r}\times\hat{\theta})\frac{\partial}{\partial \theta} + \frac{\hat{r}\times\hat{\phi}}{\sin\theta}\frac{\partial}{\partial \phi} \right]$$

$$= \frac{\hbar}{i}\left[\hat{\phi}\frac{\partial}{\partial \theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial \phi} \right]$$



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$$\overrightarrow{L} = \frac{\hbar}{i}\left[r(\hat{r}\times\hat{r})\frac{\partial}{\partial r} + (\hat{r}\times\hat{\theta})\frac{\partial}{\partial\theta} + \frac{\hat{r}\times\hat{\phi}}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

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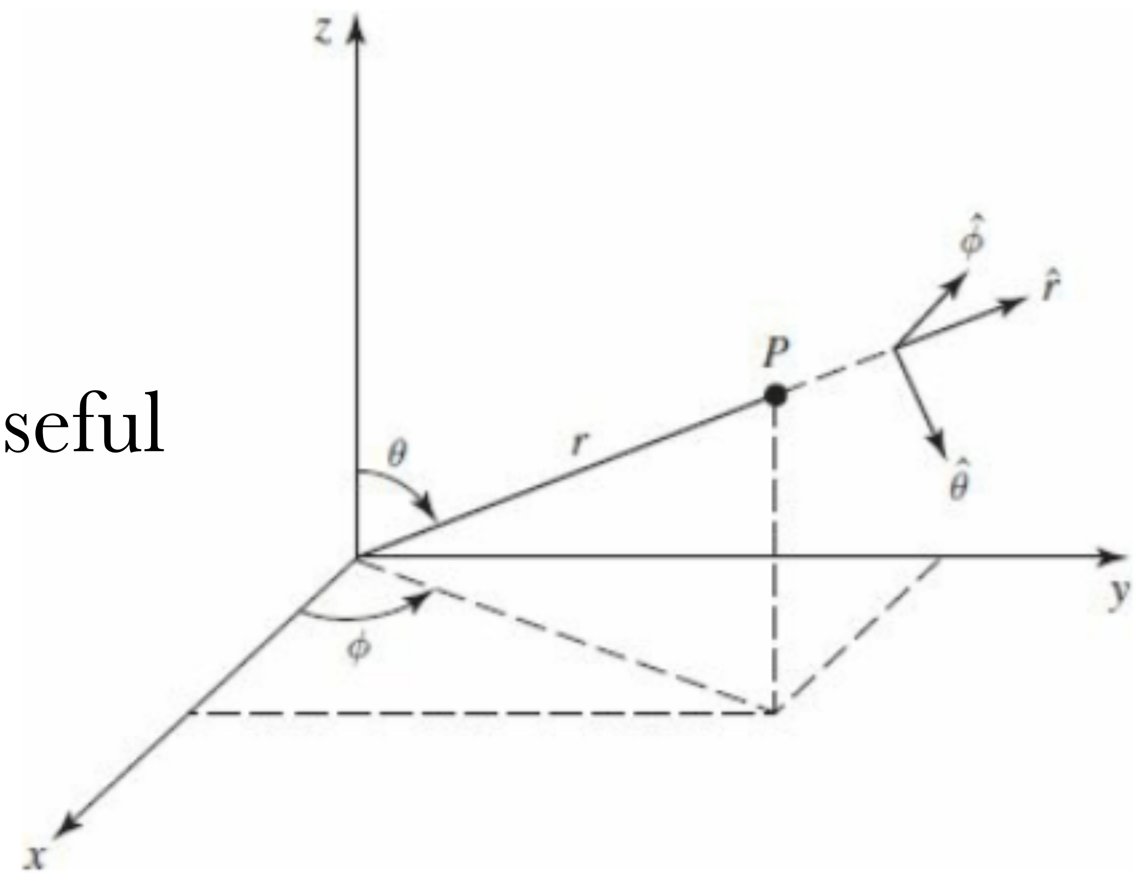
Cartesian components will also be useful



$$\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$$

$$\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$$

$$\overrightarrow{L} = \frac{\hbar}{i}\left[\hat{\phi}\frac{\partial}{\partial\theta} - \frac{\hat{\theta}}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

$$= \frac{\hbar}{i}\left[(-s\phi\hat{x} + c\phi\hat{y})\frac{\partial}{\partial\theta} - (c\theta c\phi\hat{x} + c\theta s\phi\hat{y} - s\theta\hat{z})\frac{1}{s\theta}\frac{\partial}{\partial\phi}\right]$$

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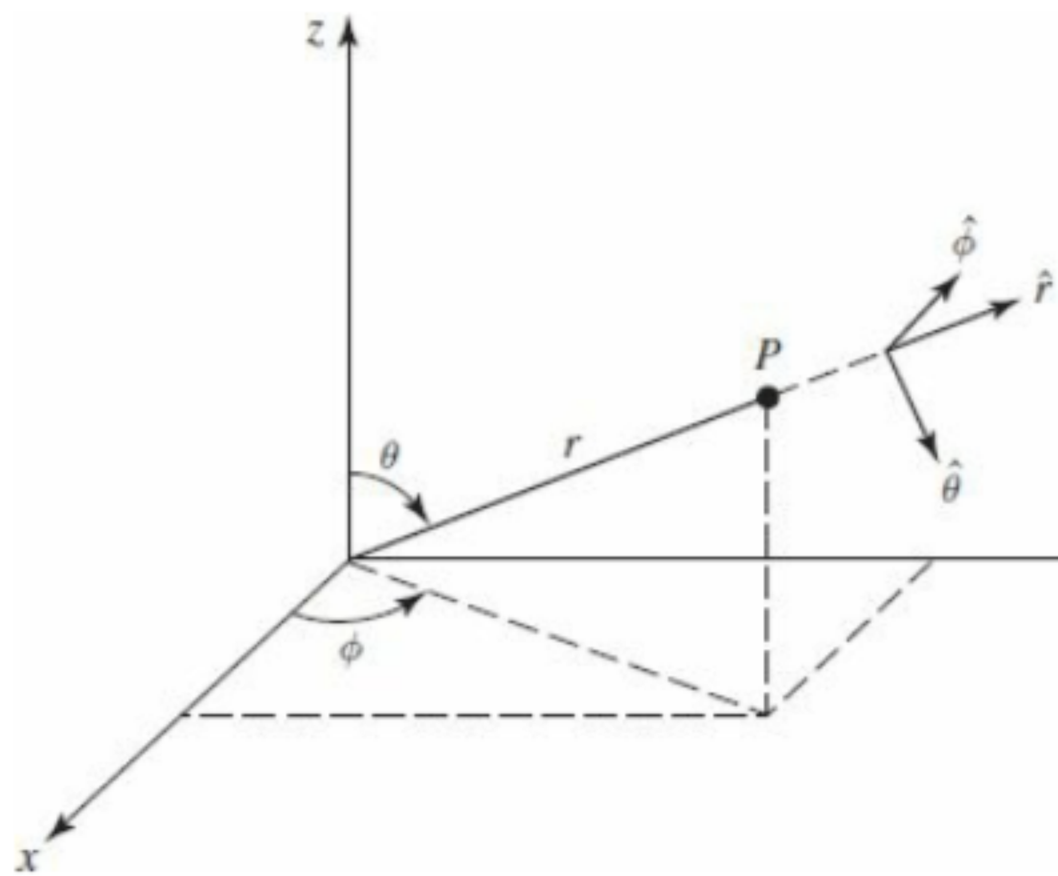
Then,

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

while

$$L_x = \frac{\hbar}{i}\left(-s\phi\frac{\partial}{\partial\theta} - \cot\theta c\phi\frac{\partial}{\partial\phi}\right)$$

$$L_y = \frac{\hbar}{i}\left(c\phi\frac{\partial}{\partial\theta} - \cot\theta s\phi\frac{\partial}{\partial\phi}\right)$$



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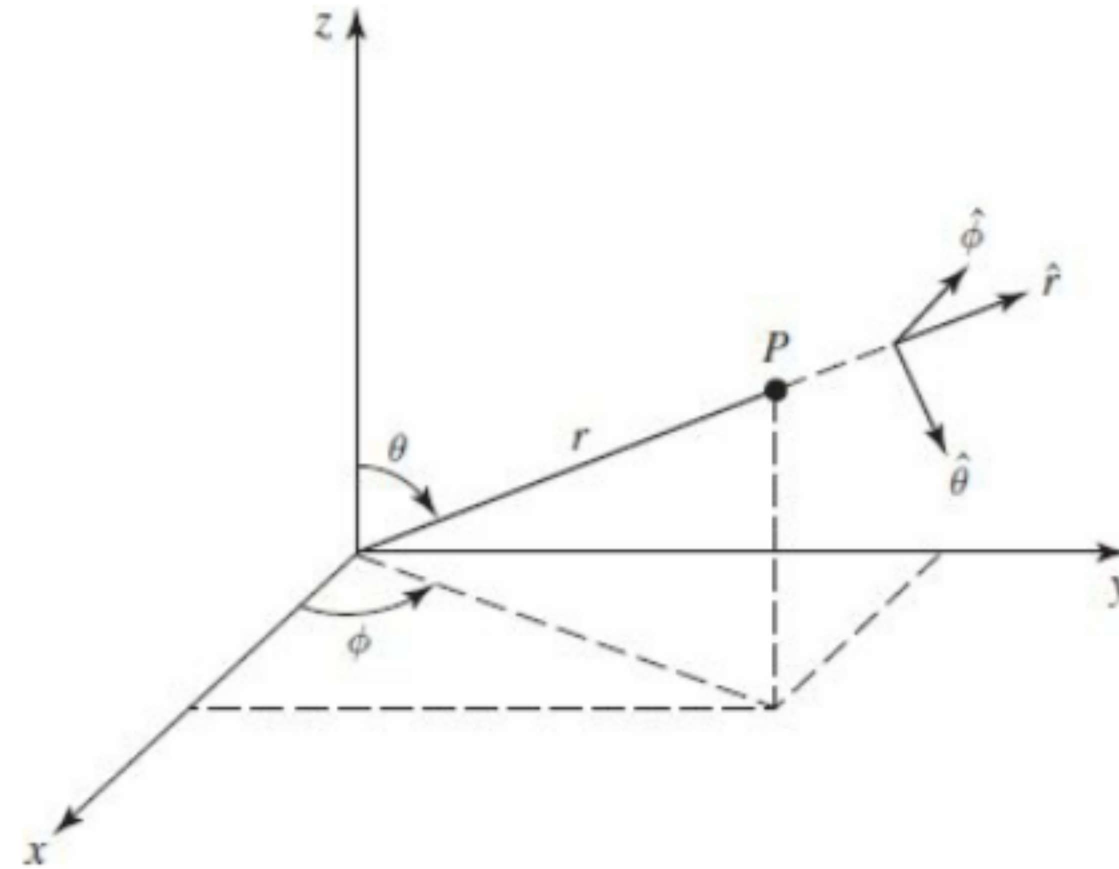

The ladder operator are slightly simpler

$$L_\pm = L_x \pm iL_y = \frac{\hbar}{i}\left[(-s\phi \pm ic\phi)\frac{\partial}{\partial\theta} - (c\phi \pm is\phi)\cot\theta\frac{\partial}{\partial\phi}\right]$$

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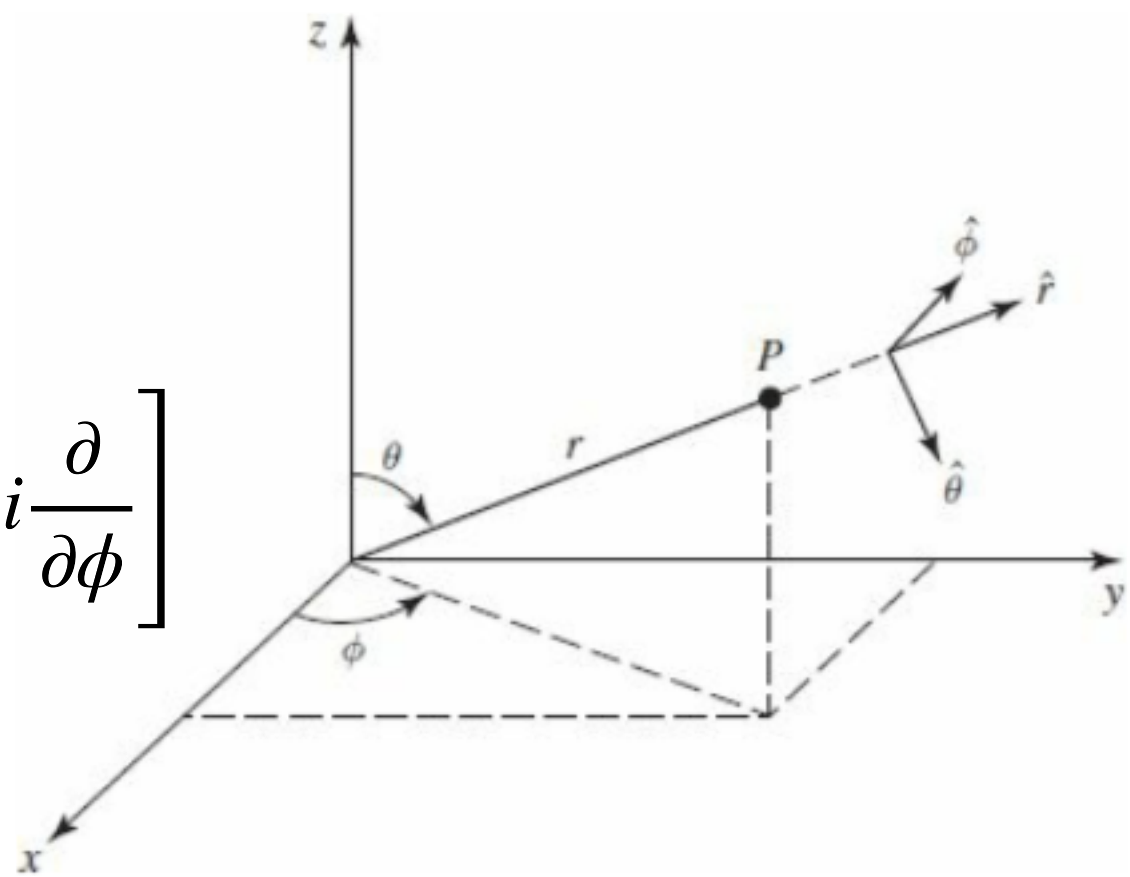

From which

$$L_+L_- = -\,\hbar^2\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right]$$

$$L^2 = -\,\hbar^2\left[\frac{1}{s\theta}\frac{\partial}{\partial\theta}\left(s\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{s^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Now, recall

$L^2 f_\ell^m = \hbar^2\ell(\ell+1)f_\ell^m$, where $f_\ell^m = f_\ell^m(\theta,\phi)$.

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Now, recall

$L^2 f_\ell^m = \hbar^2\ell(\ell+1)f_\ell^m$, where $f_\ell^m = f_\ell^m(\theta,\phi)$. Let's suppose

$f_\ell^m(\theta,\phi) = \Theta(\theta)\Phi(\phi)$ (separation ansatz)

to find

$$-\hbar^2\left[\frac{1}{s\theta}\frac{\partial}{\partial\theta}\left(s\theta\Phi\frac{d\Theta}{d\theta}\right) + \frac{1}{s^2\theta}\Theta\frac{d^2\Phi}{d\phi^2}\right] = \hbar^2\ell(\ell+1)\Theta\Phi\,.$$

Dividing by $\Theta\Phi$ and multiplying by $-s^2\theta$ gives

$$\frac{1}{\Theta}\left[s\theta\frac{d}{d\theta}\left(s\theta\frac{d\Theta}{d\theta}\right)\right] + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\ell(\ell+1)s^2\theta$$

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Dividing by $\Theta\Phi$ and multiplying by $-s^2\theta$ gives

$$\frac{1}{\Theta}\left[s\theta\frac{d}{d\theta}\left(s\theta\frac{d\Theta}{d\theta}\right)\right]+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-\ell(\ell+1)s^2\theta$$

Rearranging gives

$$\left\{\frac{1}{\Theta}\left[s\theta\frac{d}{d\theta}\left(s\theta\frac{d\Theta}{d\theta}\right)\right]+\ell(\ell+1)s^2\theta\right\}+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0.$$

We've separated variables, so the contents of the curly braces and the other term are each constant. So that

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-m^2 \quad (1)$$

$$\left\{\frac{1}{\Theta}\left[s\theta\frac{d}{d\theta}\left(s\theta\frac{d\Theta}{d\theta}\right)\right]+\ell(\ell+1)s^2\theta\right\}=m^2\ (2).$$