Today

I. Last Time

II. Summary of Results on Hydrogen

III. A Second Look at Spin

IV. Addition of Angular Momenta

I. Last time

*Introduced spin, e.g. algebraically it has the exact same structure as orbital angular momentum: $[\mathsf{S}_x, \mathsf{S}_y] = i\hbar \mathsf{S}_z$.

*Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the spin operators are then just

$$\mathsf{S}_\mathsf{i} = \frac{\hbar}{2}\sigma_i.$$

*Found the radial wave functions for the Hydrogen using the power series method for solving differential equaitons.

II. Series Method for Hydrogen atom

Then the total wave function is

$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)Y_\ell^m(\theta,\phi)$,

Where

$R_{n\ell}(r) = \frac{1}{r}\rho^{\ell+1}e^{-\rho}v(\rho)$,

Where $v(\rho)$ is determined by

$$c_{j+1} = \frac{2(j+\ell+1-n)}{(j+1)(j+2\ell+2)}c_j\,.$$

Also $\rho \equiv \frac{r}{an}$.

$$E_n = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2}$$

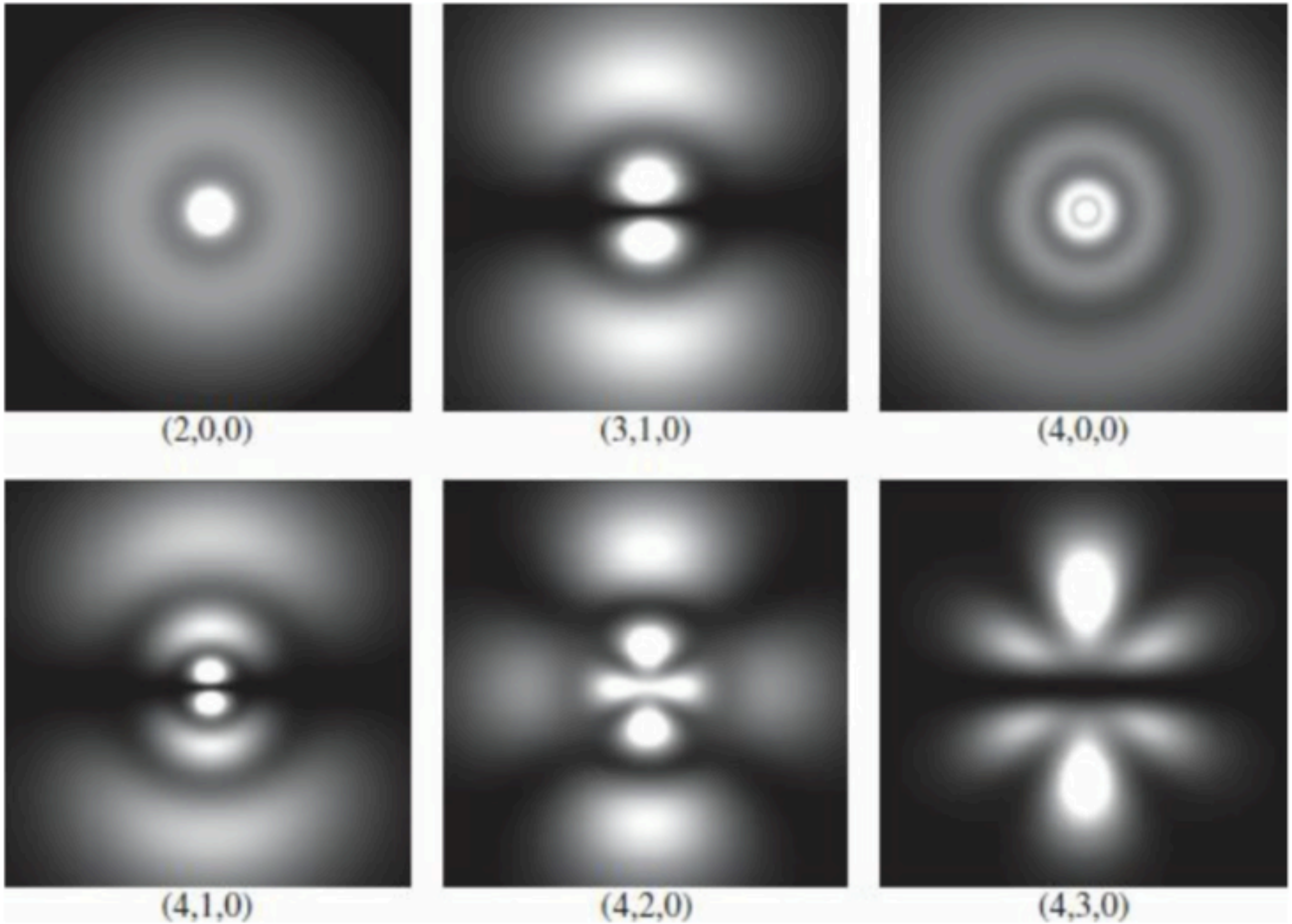
$\frac{1}{\lambda} = \mathscr{R}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$, with

$$\mathscr{R} \equiv \frac{m_e}{4\pi c\hbar^3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = 1.097\times 10^7\mathrm{m}^{-1}$$

Alternatively we can write these things as

$$\psi_{n\ell m} = \sqrt{\left(\frac{2}{na}\right)^3\frac{(n-\ell-1)!}{2n(n+\ell)!}}e^{-r/na}\left(\frac{2r}{na}\right)^\ell\left[L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na}\right)\right]Y_\ell^m(\theta,\phi)$$

II. Hydrogen atom wave functions



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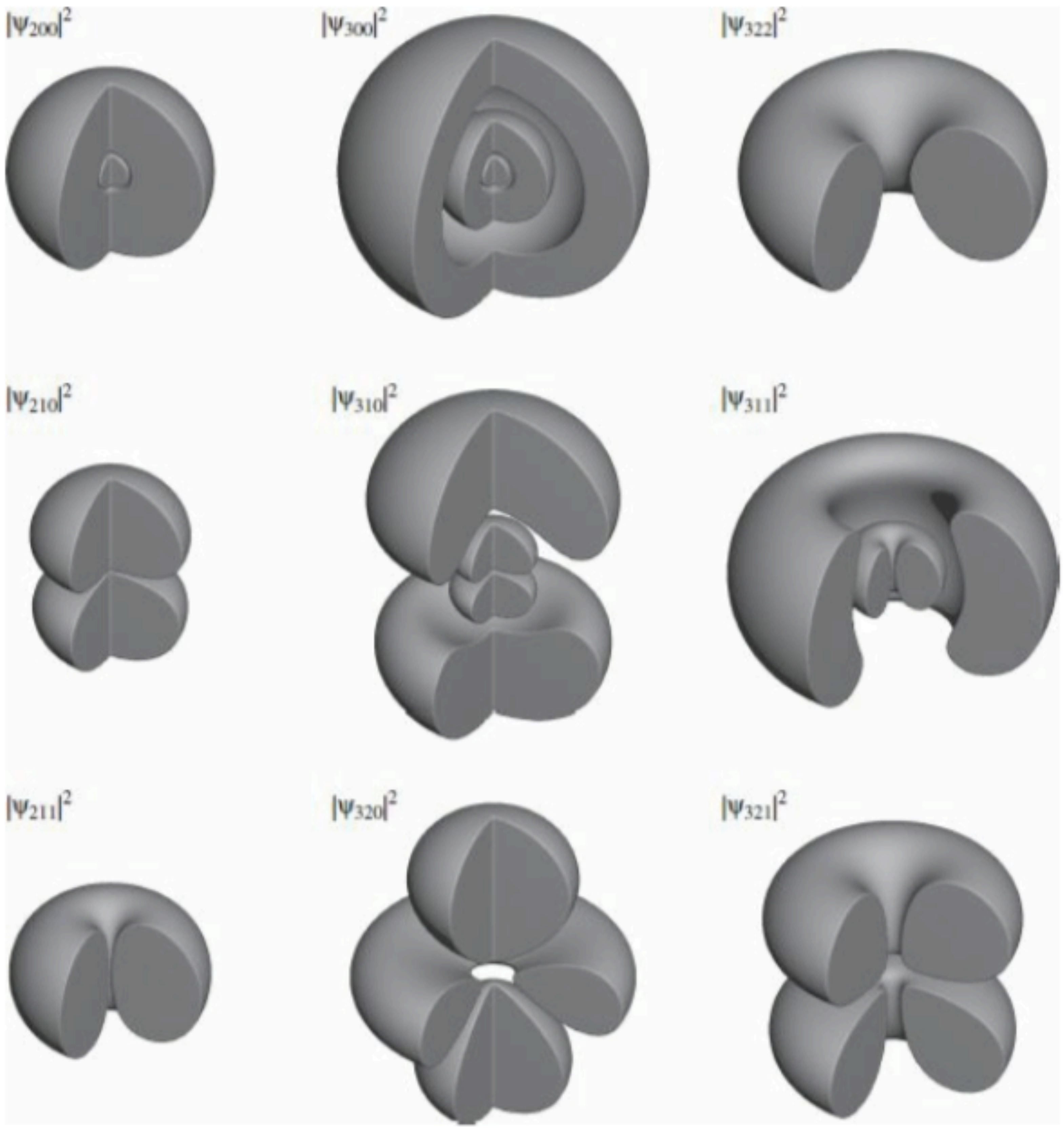
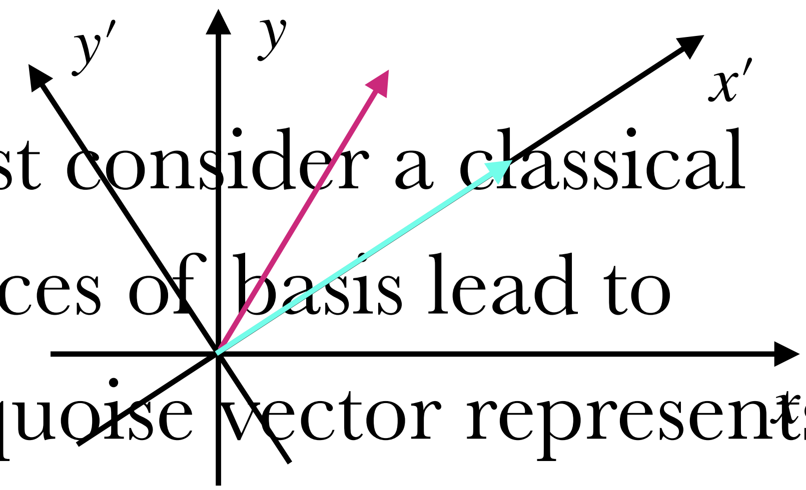


III. A Second Look at Spin



Thinking about bases can be subtle. Let's first consider a classical vector (the one purple above). Different choices of basis lead to different expressions for this vector. The turquoise vector represents the basis vector $\hat{x}'$, which I can express in the xy-basis.

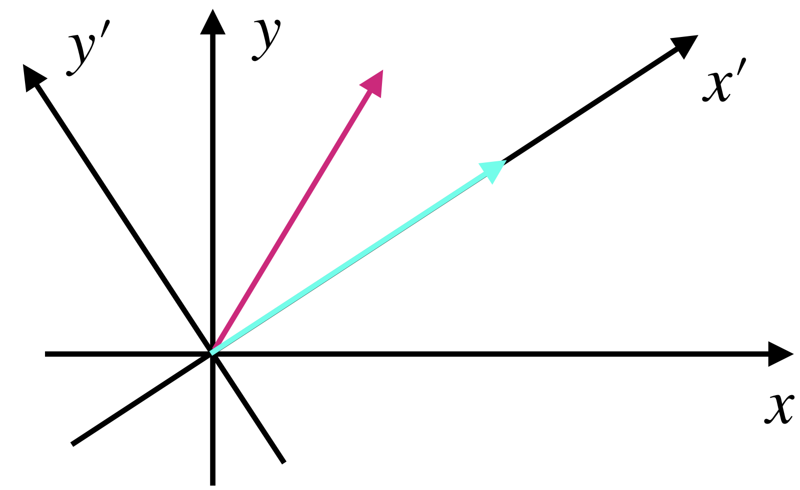
First let's relate Dirac notation to Matrix notation:

$|s\ m\rangle$ or more specifically $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle \equiv |\uparrow\rangle \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$ (in the z-basis),

The spin 1/2 particle down is $\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle \equiv |\downarrow\rangle \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$.

In general, a spin state is a general superposition of these two states, $\chi = \begin{pmatrix}a\\b\end{pmatrix} = a\chi_+ + b\chi_-$; in this state the probability of measuring spin up is $|a|^2$ and that of spin down is $|b|^2$.

III. A Second Look at Spin

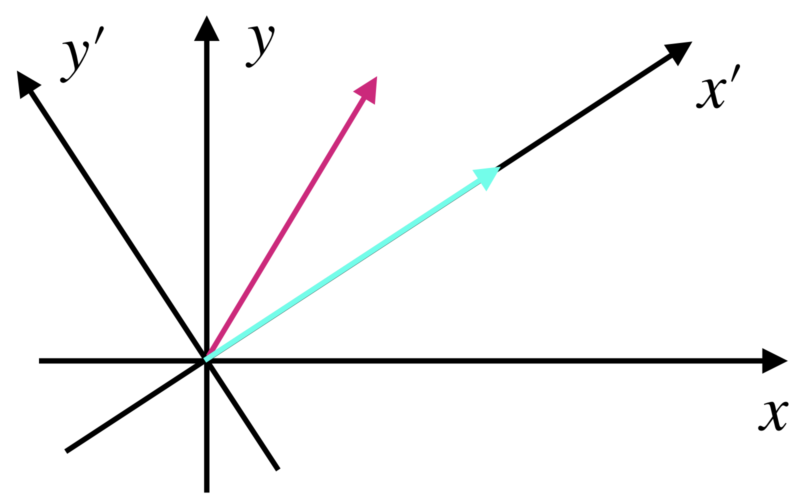


Saiqi introduced two new states $\chi_+^{(x)}$ and $\chi_-^{(x)}$, which represent spin up in the x-direction and spin down in the x-direction. Then he proved that these two states can be expressed in the usual z-basis. The results that he found were

$$\chi_+^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{and} \quad \chi_-^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

If I prepare the quantum state of a spinning particle in the states $\chi_\pm^{(x)}$, then measurements of the spin of that particle along the z-axis will give us spin up and spin down with equal probabilities of 50%.

III. A Second Look at Spin



Suppose we prepared the state

$\chi_-^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, then what would a

measurement of spin in the y-direction give us? First, what are the possible values? Compute the eigenvalues of S_y and it turns out that they are $\pm\frac{\hbar}{2}$. To check Zak's claim that these outcomes are equal probability

we first have to express $\chi_-^{(x)}$ as a superposition of y eigenstates as $\chi_-^{(x)} = a\chi_+^{(y)} + b\chi_-^{(y)}$, then probability of getting spin up in the y-direction is $|a|^2$, and spin down in the y-direction is $|b|^2$.

IV. Addition of Angular Momentum

We transition back to Dirac notation to describe systems of two spins. The basic state of such a system is

$|s_1\ s_2\ m_1\ m_2\rangle$.

For each of these spins we have again

$S^{(1)^2}|s_1\ s_2\ m_1\ m_2\rangle = s_1(s_1+1)\hbar^2|s_1\ s_2\ m_1\ m_2\rangle$

$S^{(2)^2}|s_1\ s_2\ m_1\ m_2\rangle = s_2(s_2+1)\hbar^2|s_1\ s_2\ m_1\ m_2\rangle$

$S_z^{(1)}|s_1\ s_2\ m_1\ m_2\rangle = m_1\hbar|s_1\ s_2\ m_1\ m_2\rangle$

$S_z^{(2)}|s_1\ s_2\ m_1\ m_2\rangle = m_2\hbar|s_1\ s_2\ m_1\ m_2\rangle$.

What is the total spin angular momentum of this system?

$\overrightarrow{S} = \overrightarrow{S}^{(1)} + \overrightarrow{S}^{(2)}$.

Now eigenvalues. The z-component isn't bad

$S_z|s_1\ s_2\ m_1\ m_2\rangle = S_z^{(1)}|s_1\ s_2\ m_1\ m_2\rangle + S_z^{(2)}|s_1\ s_2\ m_1\ m_2\rangle = \hbar(m_1+m_2)|s_1\ s_2\ m_1\ m_2\rangle$

The total angular momentum quantum number s is more subtle. It leads into the story of Clebsch-Gordan coefficients.