Today

I. Last Time

II. Addition of Angular Momenta

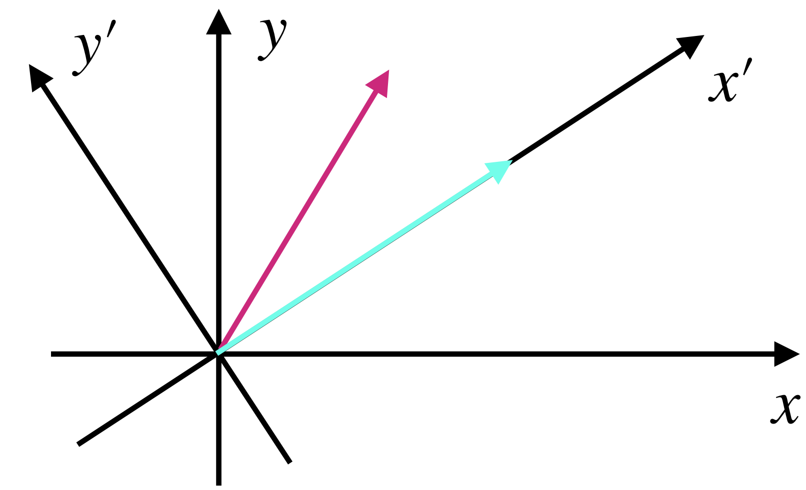
I. Last time

*Two particle system: we introduced basis states $|s_1\ s_2\ m_1\ m_2\rangle$. E.g. $\left|\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\right\rangle \equiv |\uparrow\uparrow\rangle$. The total z-angular momentum operator was defined by: $S_z = S_z^{(1)} + S_z^{(2)}$, or more generally $\vec{S} = \vec{S}^{(1)} + \vec{S}^{(2)}$. Then $S_z|s_1\ s_2\ m_1\ m_2\rangle = (m_1 + m_2)\hbar|s_1\ s_2\ m_1\ m_2\rangle$.

[Aside: Given the state $|s\ m\rangle$, the m quantum number ranges from $-s$ to s in integer steps.]

*A spin that is in a definite state of z-angular momentum, is also in a mixed state of x-angular momentum. In particular, we showed that a spin up in the z-direction particle is an equal mixture of x spins.

III. A Second Look at Spin



Thinking about bases can be subtle. Let's first consider a classical vector (the one purple above). Different choices of basis lead to different expressions for this vector. The turquoise vector represents the basis vector $\hat{x}'$, which I can express in the xy-basis.

First let's relate Dirac notation to Matrix notation:

$|s\ m\rangle$ or more specifically $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle \equiv |\uparrow\rangle \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix}$ (in the z-basis),

The spin 1/2 particle down is $\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle \equiv |\downarrow\rangle \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$.

In general, a spin state is a general superposition of these two states, $\chi = \begin{pmatrix}a\\b\end{pmatrix} = a\chi_+ + b\chi_-$; in this state the probability of measuring spin up is $|a|^2$ and that of spin down is $|b|^2$.

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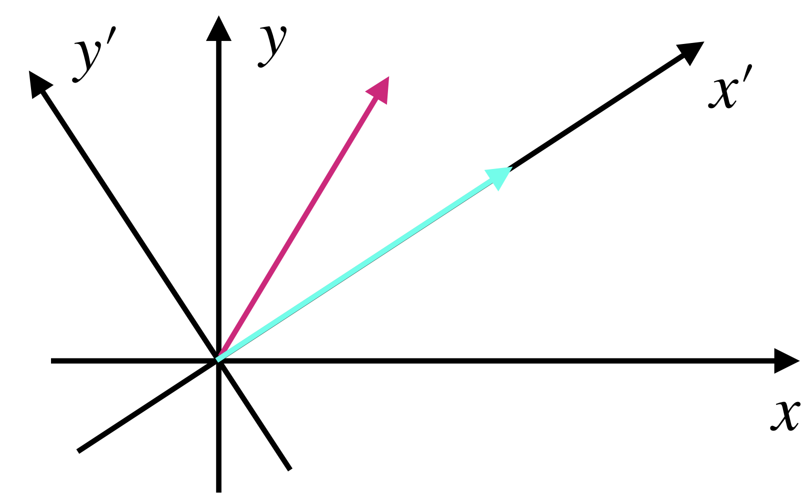

Saiqi introduced two new states $\chi_+^{(x)}$ and $\chi_-^{(x)}$, which represent spin up in the x-direction and spin down in the x-direction. Then he proved that these two states can be expressed in the usual z-basis. The results that he found were

$$\chi_+^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{and} \quad \chi_-^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

If I prepare the quantum state of a spinning particle in the states $\chi_\pm^{(x)}$, then measurements of the spin of that particle along the z-axis will give us spin up and spin down with equal probabilities of 50%.

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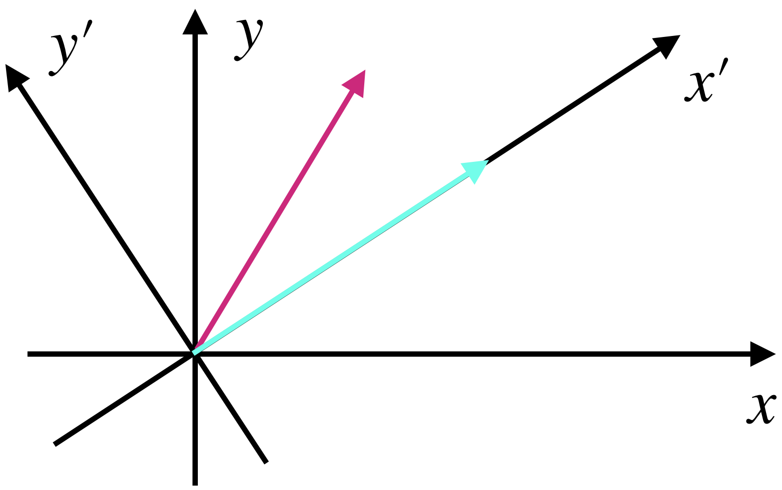

Suppose we prepared the state

$$\chi_-^{(x)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix},$$ then what would a

measurement of spin in the y-direction give us? First, what are the possible values? Compute the eigenvalues of S_y and it turns out that they are $\pm\frac{\hbar}{2}$. To check Zak's claim that these outcomes are equal probability

we first have to express $\chi_-^{(x)}$ as a superposition of y eigenstates as $\chi_-^{(x)} = a\chi_+^{(y)} + b\chi_-^{(y)}$, then probability of getting spin up in the y -direction is $|a|^2$, and spin down in the y-direction is $|b|^2$.

IV. Addition of Angular Momentum

We transition back to Dirac notation to describe systems of two spins. The basic state of such a system is

$|s_1\ s_2\ m_1\ m_2\rangle.$

For each of these spins we have again

$S^{(1)^2}|s_1\ s_2\ m_1\ m_2\rangle = s_1(s_1+1)\hbar^2|s_1\ s_2\ m_1\ m_2\rangle$

$S^{(2)^2}|s_1\ s_2\ m_1\ m_2\rangle = s_2(s_2+1)\hbar^2|s_1\ s_2\ m_1\ m_2\rangle$

$S_z^{(1)}|s_1\ s_2\ m_1\ m_2\rangle = m_1\hbar|s_1\ s_2\ m_1\ m_2\rangle$

$S_z^{(2)}|s_1\ s_2\ m_1\ m_2\rangle = m_2\hbar|s_1\ s_2\ m_1\ m_2\rangle.$

What is the total spin angular momentum of this system?

$\overrightarrow{S} = \overrightarrow{S}^{(1)} + \overrightarrow{S}^{(2)}.$

Now eigenvalues. The z-component isn't bad

$S_z|s_1\ s_2\ m_1\ m_2\rangle = S_z^{(1)}|s_1\ s_2\ m_1\ m_2\rangle + S_z^{(2)}|s_1\ s_2\ m_1\ m_2\rangle = \hbar(m_1+m_2)|s_1\ s_2\ m_1\ m_2\rangle$

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II. Addition of Angular Momentum

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The total angular momentum quantum number s is more subtle. It leads into the story of Clebsch-Gordan coefficients.

Let's consider the specific case of two spin 1/2 particles:

$|\uparrow\uparrow\rangle = \left|\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\right\rangle$, with $m = 1$

$|\uparrow\downarrow\rangle$, with $m = 0$

$|\downarrow\uparrow\rangle$, with $m = 0$

$|\downarrow\downarrow\rangle$, with $m = -1$.

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Where are the $s = 1$ states? To answer this, let's try using the lowering operator on the state $|\uparrow\uparrow\rangle$. The lowering operator is $S_- = S_-^{(1)} + S_-^{(2)}$, and so we find

$S_-|\uparrow\uparrow\rangle = \left(S_-^{(1)}|\uparrow\rangle\right)|\uparrow\rangle + |\uparrow\rangle\left(S_-^{(2)}|\uparrow\rangle\right) = \left(\hbar|\downarrow\rangle\right)|\uparrow\rangle + |\uparrow\rangle\left(\hbar|\downarrow\rangle\right)$

$= \hbar(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$. Acting S_- again gives you $|\downarrow\downarrow\rangle$.

II. Addition of Angular Momentum

We've just identified a triplet of states

$$\begin{cases} |1\ 1\rangle = |\uparrow\uparrow\rangle, s = 1 \text{ and } m = 1, \\ |1\ 0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle),\ s = 1 \text{ and } m = 0, \\ |1\ -1\rangle = |\downarrow\downarrow\rangle,\ s = 1 \text{ and } m = -1 \end{cases}$$

These are a basis for the states with $|s = 1\ m\rangle$. We can construct a fourth state by orthogonality:

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

$$\frac{1}{2}(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) = \frac{1}{2}(0 + 1 - 1 - 0) = 0.$$

This new 'singlet' state is

$|0\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, this is the $s = 0$ (singlet)

II. Addition of Angular Momentum

Handle with care! Here's a classical analog:

$$S^2 | s \; m \rangle = s(s+1)\hbar^2 | s \; m \rangle$$