Today

I. Last Time

II. Last words on Addition of Angular Momenta

III. Zak's Guest Lecture on Two Particle Systems

IV. Spatial and Spin Wave Functions

I. Last time

* Studied the ways in which we could achieve $s = 1$ by combining two spin 1/2 particles. This gives the triplet

$$|1\ m\rangle = \begin{cases} |\uparrow\uparrow\rangle, \text{gives } m = 1 \\ \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \text{gives } m = 0 \\ |\downarrow\downarrow\rangle, \text{gives } m - 1 \end{cases},$$

We also found the singlet state

$$|0\ 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

I. Addition of Angular Momentum

We transition back to Dirac notation to describe systems of two spins. The basic state of such a system is

$|s_1\ s_2\ m_1\ m_2\rangle.$

For each of these spins we have again

$S^{(1)^2}|s_1\ s_2\ m_1\ m_2\rangle = s_1(s_1+1)\hbar^2|s_1\ s_2\ m_1\ m_2\rangle$

$S^{(2)^2}|s_1\ s_2\ m_1\ m_2\rangle = s_2(s_2+1)\hbar^2|s_1\ s_2\ m_1\ m_2\rangle$

$S_z^{(1)}|s_1\ s_2\ m_1\ m_2\rangle = m_1\hbar|s_1\ s_2\ m_1\ m_2\rangle$

$S_z^{(2)}|s_1\ s_2\ m_1\ m_2\rangle = m_2\hbar|s_1\ s_2\ m_1\ m_2\rangle.$

What is the total spin angular momentum of this system?

$\overrightarrow{S} = \overrightarrow{S}^{(1)} + \overrightarrow{S}^{(2)}.$

Now eigenvalues. The z-component isn't bad

$S_z|s_1\ s_2\ m_1\ m_2\rangle = S_z^{(1)}|s_1\ s_2\ m_1\ m_2\rangle + S_z^{(2)}|s_1\ s_2\ m_1\ m_2\rangle = \hbar(m_1+m_2)|s_1\ s_2\ m_1\ m_2\rangle$

The total angular momentum quantum number s is more subtle. It leads into the story of Clebsch-Gordan coefficients.

I. Addition of Angular Momentum

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The total angular momentum quantum number s is more subtle. It leads into the story of Clebsch-Gordan coefficients.

Let's consider the specific case of two spin 1/2 particles:

$|\uparrow\uparrow\rangle = \left|\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\right\rangle$, with $m = 1$

$|\uparrow\downarrow\rangle$, with $m = 0$

$|\downarrow\uparrow\rangle$, with $m = 0$

$|\downarrow\downarrow\rangle$, with $m = -1$.

Where are the $s = 1$ states?

I. Addition of Angular Momentum

Let's consider the specific case of two spin 1/2 particles:

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$|\uparrow\downarrow\rangle$, with $m = 0$

$|\downarrow\uparrow\rangle$, with $m = 0$

$|\downarrow\downarrow\rangle$, with $m = -1$.

Where are the $s = 1$ states? To answer this, let's try using the lowering operator on the state $|\uparrow\uparrow\rangle$. The lowering operator is $S_- = S_-^{(1)} + S_-^{(2)}$, and so we find

$$S_-|\uparrow\uparrow\rangle = \left(S_-^{(1)}|\uparrow\rangle\right)|\uparrow\rangle + |\uparrow\rangle\left(S_-^{(2)}|\uparrow\rangle\right) = \left(\hbar|\downarrow\rangle\right)|\uparrow\rangle + |\uparrow\rangle\left(\hbar|\downarrow\rangle\right)$$

$= \hbar(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle)$. Acting S_- again gives you $|\downarrow\downarrow\rangle$.

I. Addition of Angular Momentum

We've just identified a triplet of states

$$\begin{cases} |1\ 1\rangle = |\uparrow\uparrow\rangle, s = 1 \text{ and } m = 1, \\ |1\ 0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle),\ s = 1 \text{ and } m = 0, \\ |1\ -1\rangle = |\downarrow\downarrow\rangle,\ s = 1 \text{ and } m = -1 \end{cases}$$

These are a basis for the states with $|s = 1\ m\rangle$. We can construct a fourth state by orthogonality:

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

$$\frac{1}{2}(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) = \frac{1}{2}(0 + 1 - 1 - 0) = 0.$$

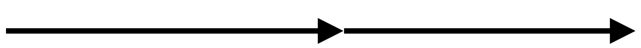
This new 'singlet' state is

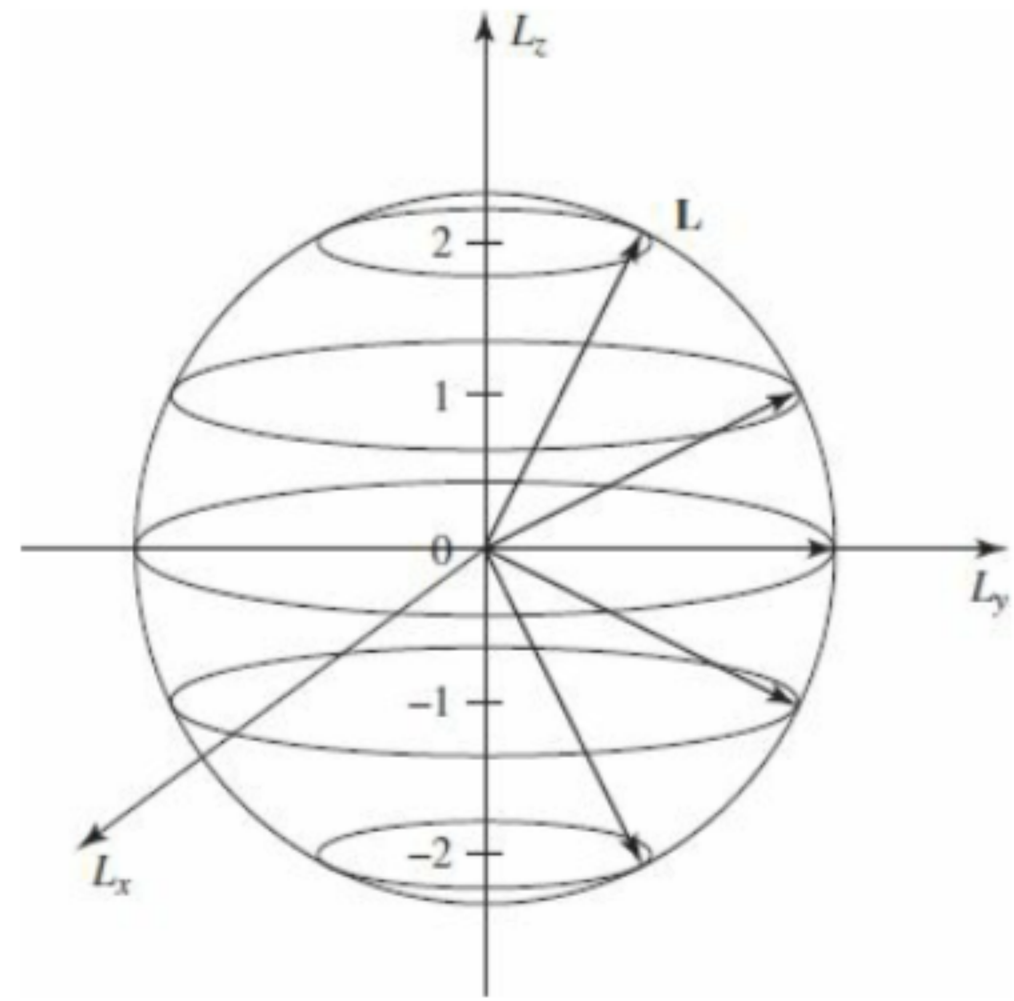
$|0\ 0\rangle = \dfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, this is the $s = 0$ (singlet)

II. Addition of Angular Momentum

Handle with care! Here’s a classical analog:

$$S^2 |s\ m\rangle = s(s+1)\hbar^2 |s\ m\rangle$$





II. Final words on Addition of Angular Momentum

Suppose we now wanted to add up two general spins s_1 and s_2, what possible results would we get? The strategy, which we won't carry out, is the same as in our example from last time: you pick the highest spin state and act lowering operators to construct all the intermediate states. The result is that you can get any spin in the following list:

$s \in \{s_1 + s_2, s_1 + s_2 - 1, s_1 + s_2 - 2, \ldots, |s_1 - s_2|\}.$

In fact, this holds for the addition of any two angular momenta,

$j \in \{s + \ell, s + \ell - 1, \ldots, |\ell - s|\}.$

This is all summarized in so-called Clebsch-Gordan coefficients:

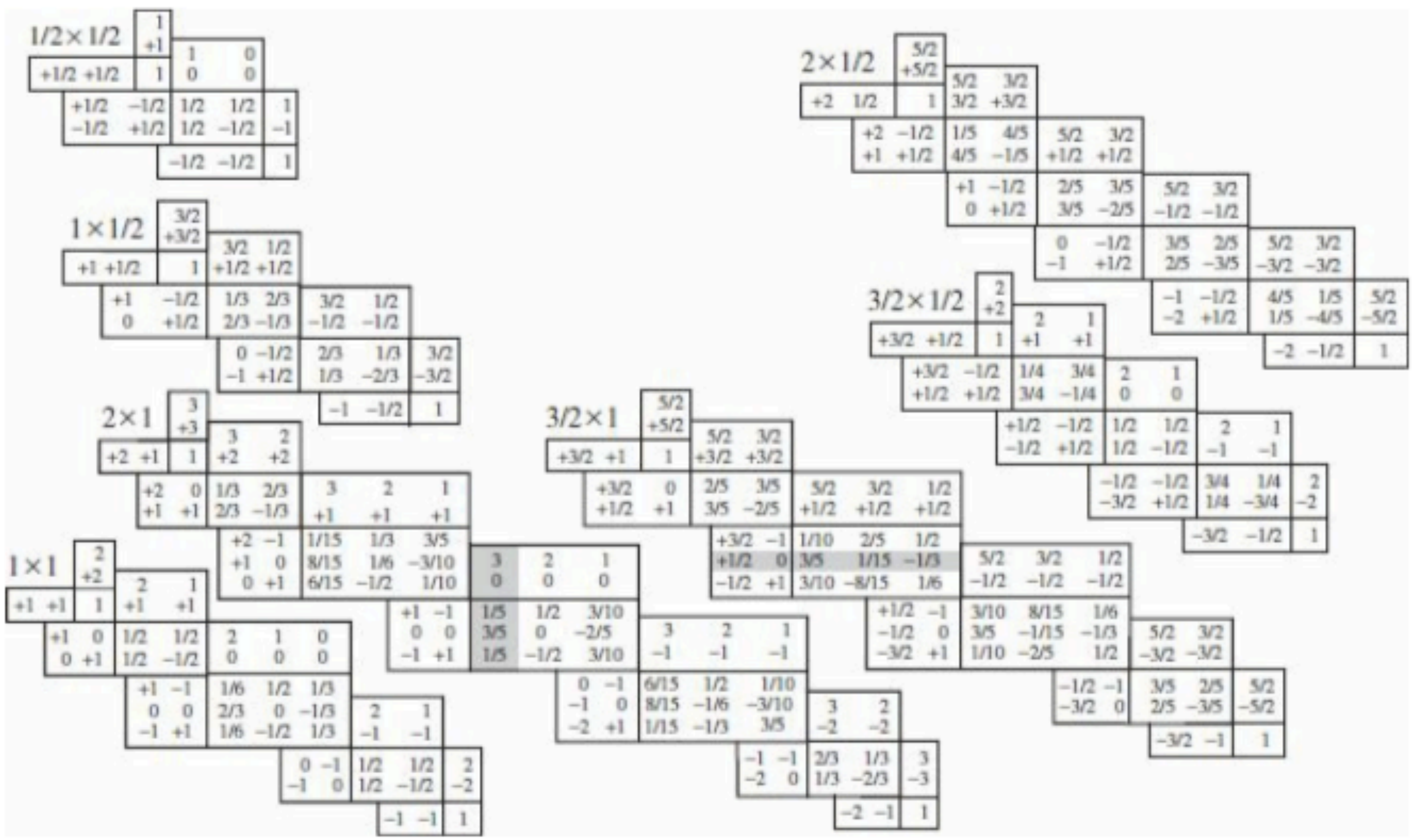
$$|s\ m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1\ s_2\ m_1\ m_2\rangle$$

II. Final words on Addition of Angular Momentum

This is all summarized in so-called Clebsch-Gordan coefficients:

$$|s\ m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m}\,|s_1\ s_2\ m_1\ m_2\rangle.$$

Here's a small section of such a table



1/2 × 1/2

m_1 m_2	1, +1
+1/2 +1/2	1

m_1 m_2	1, 0	0, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	1, −1
−1/2 −1/2	1

1 × 1/2

m_1 m_2	3/2, +3/2
+1 +1/2	1

m_1 m_2	3/2, +1/2	1/2, +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	3/2, −1/2	1/2, −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	3/2, −3/2
−1 −1/2	1

2 × 1

m_1 m_2	3, +3
+2 +1	1

m_1 m_2	3, +2	2, +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	3, +1	2, +1	1, +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	6/15	−1/2	1/10

m_1 m_2	3, 0	2, 0	1, 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	3, −1	2, −1	1, −1
0 −1	6/15	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	3, −2	2, −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	3, −3
−2 −1	1

1 × 1

m_1 m_2	2, +2
+1 +1	1

m_1 m_2	2, +1	1, +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	2, 0	1, 0	0, 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	2, −1	1, −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	2, −2
−1 −1	1

3/2 × 1

m_1 m_2	5/2, +5/2
+3/2 +1	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	5/2, −3/2	3/2, −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	5/2, −5/2
−3/2 −1	1

2 × 1/2

m_1 m_2	5/2, +5/2
+2 1/2	1

m_1 m_2	5/2, 3/2	3/2, +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2, +1/2	3/2, +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2, −1/2	3/2, −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2, −3/2	3/2, −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2, −5/2
−2 −1/2	1

3/2 × 1/2

m_1 m_2	2, +2
+3/2 +1/2	1

m_1 m_2	2, +1	1, +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	2, 0	1, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

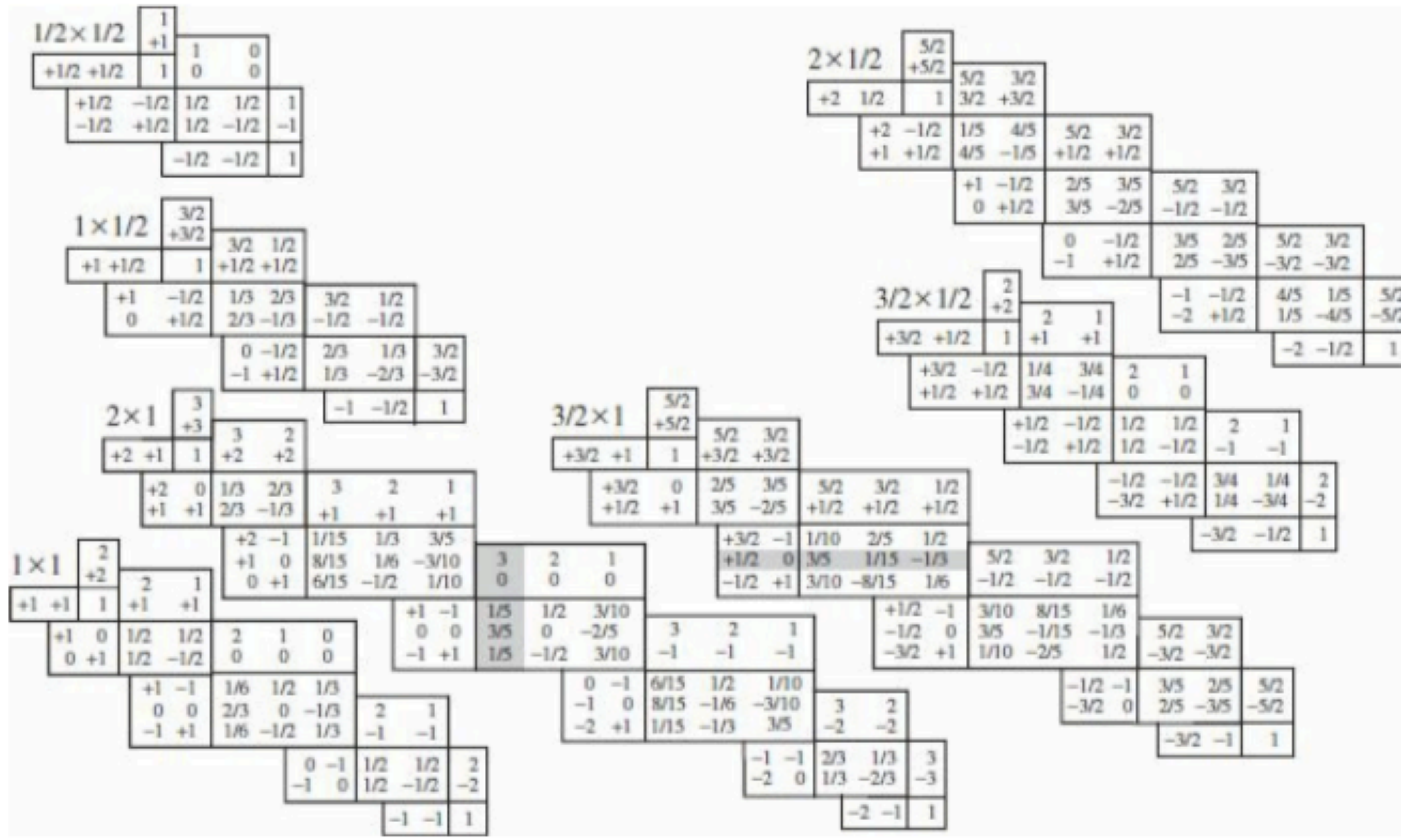
m_1 m_2	2, −1	1, −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	2, −2
−3/2 −1/2	1

As an example, consider the state $|3\ 0\rangle$ and ask how we make it out of two spins with $s_1 = 2$ and $s_2 = 1$.

II. Final words on Addition of Angular Momentum

$$|s\ m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1\ s_2\ m_1\ m_2\rangle.$$



$$|3\ 0\rangle = C_1|2\ 1\rangle|1\ \ -1\rangle + C_2|2\ 0\rangle|10\rangle + C_3|2\ \ -1\rangle|1\ 1\rangle$$

$$= \frac{1}{\sqrt{5}}|2\ 1\rangle|1\ \ -1\rangle + \sqrt{\frac{3}{5}}|2\ 0\rangle|10\rangle + \frac{1}{\sqrt{5}}|2\ \ -1\rangle|1\ 1\rangle$$