Today

I. Last Time
II. Two Particle Wave Functions in General
III. Atoms: a beginning…

I. Last time

* Zak's guest lecture on two particle systems. Symmetrized wave functions describe bosons and anti-symmetrize wave functions describe fermions. The symmetry is with respect to the exchange of the two particles. This requirement only arises because quantum mechanics allows for completely identical particles.
* Studied the combination of two angular momenta in general , and in particular, learned to use Clebsch-Gordan tables.

I. Addition of Angular Momentum

We've just identified a triplet of states

$$\begin{cases} |1\ 1\rangle = |\uparrow\uparrow\rangle, s = 1 \text{ and } m = 1, \\ |1\ 0\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle),\ s = 1 \text{ and } m = 0, \\ |1\ -1\rangle = |\downarrow\downarrow\rangle,\ s = 1 \text{ and } m = -1 \end{cases}$$

These are a basis for the states with $|s = 1\ m\rangle$. We can construct a fourth state by orthogonality:

$$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).$$

$$\frac{1}{2}(\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)(|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle) = \frac{1}{2}(0 + 1 - 1 - 0) = 0.$$

This new 'singlet' state is

$|0\ 0\rangle = \dfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, this is the $s = 0$ (singlet)

I. Final words on Addition of Angular Momentum

Suppose we now wanted to add up two general spins s_1 and s_2, what possible results would we get? The strategy, which we won't carry out, is the same as in our example from last time: you pick the highest spin state and act lowering operators to construct all the intermediate states. The result is that you can get any spin in the following list:

$s \in \{s_1 + s_2, s_1 + s_2 - 1, s_1 + s_2 - 2, \ldots, |s_1 - s_2|\}$.

In fact, this holds for the addition of any two angular momenta,

$j \in \{s + \ell, s + \ell - 1, \ldots, |\ell - s|\}$.

This is all summarized in so-called Clebsch-Gordan coefficients:

$$|s\ m\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1\ s_2\ m_1\ m_2\rangle$$

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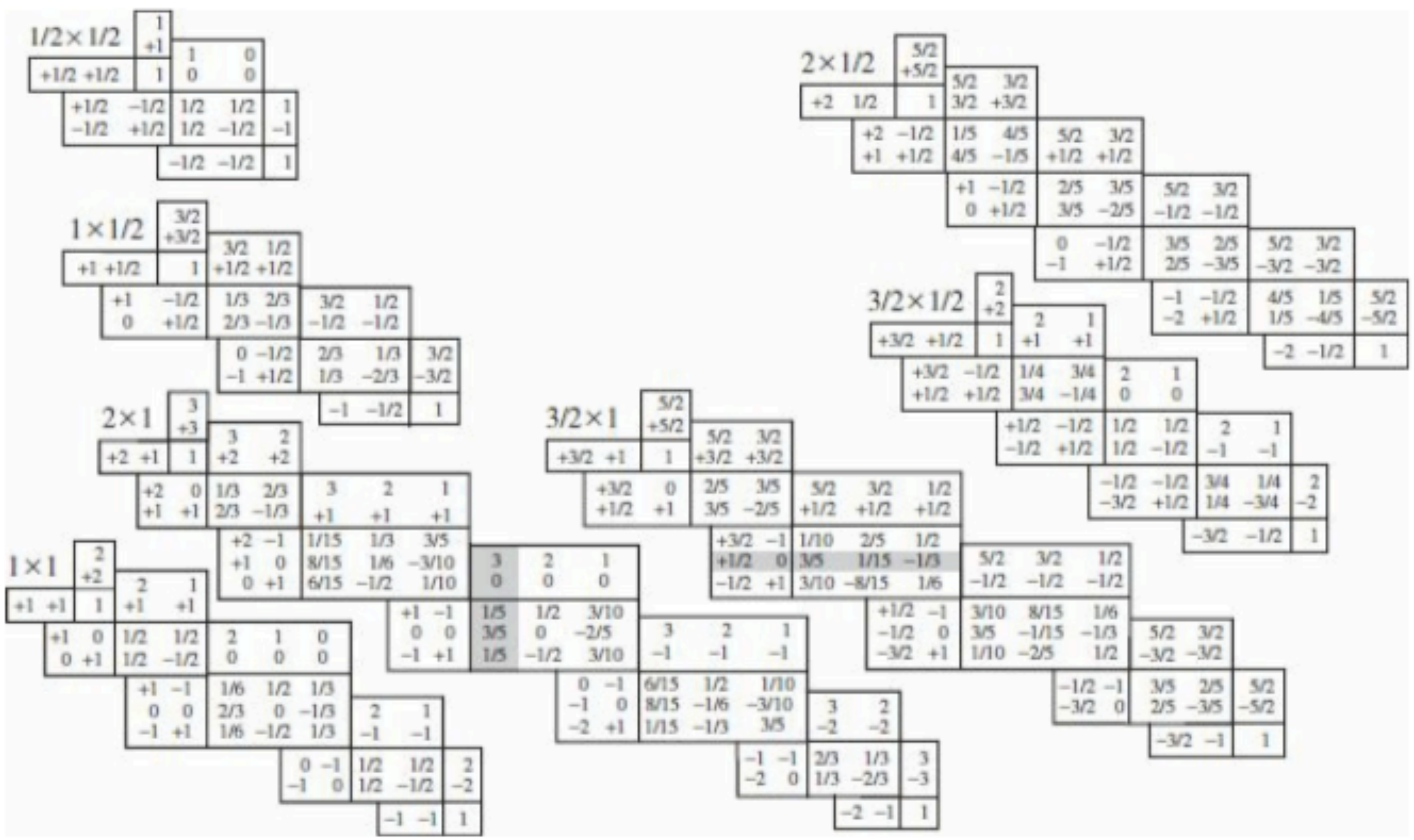
Here's a small section of such a table



As an example, consider the state $|3\ 0\rangle$ and ask how we make it out of two spins with $s_1 = 2$ and $s_2 = 1$.

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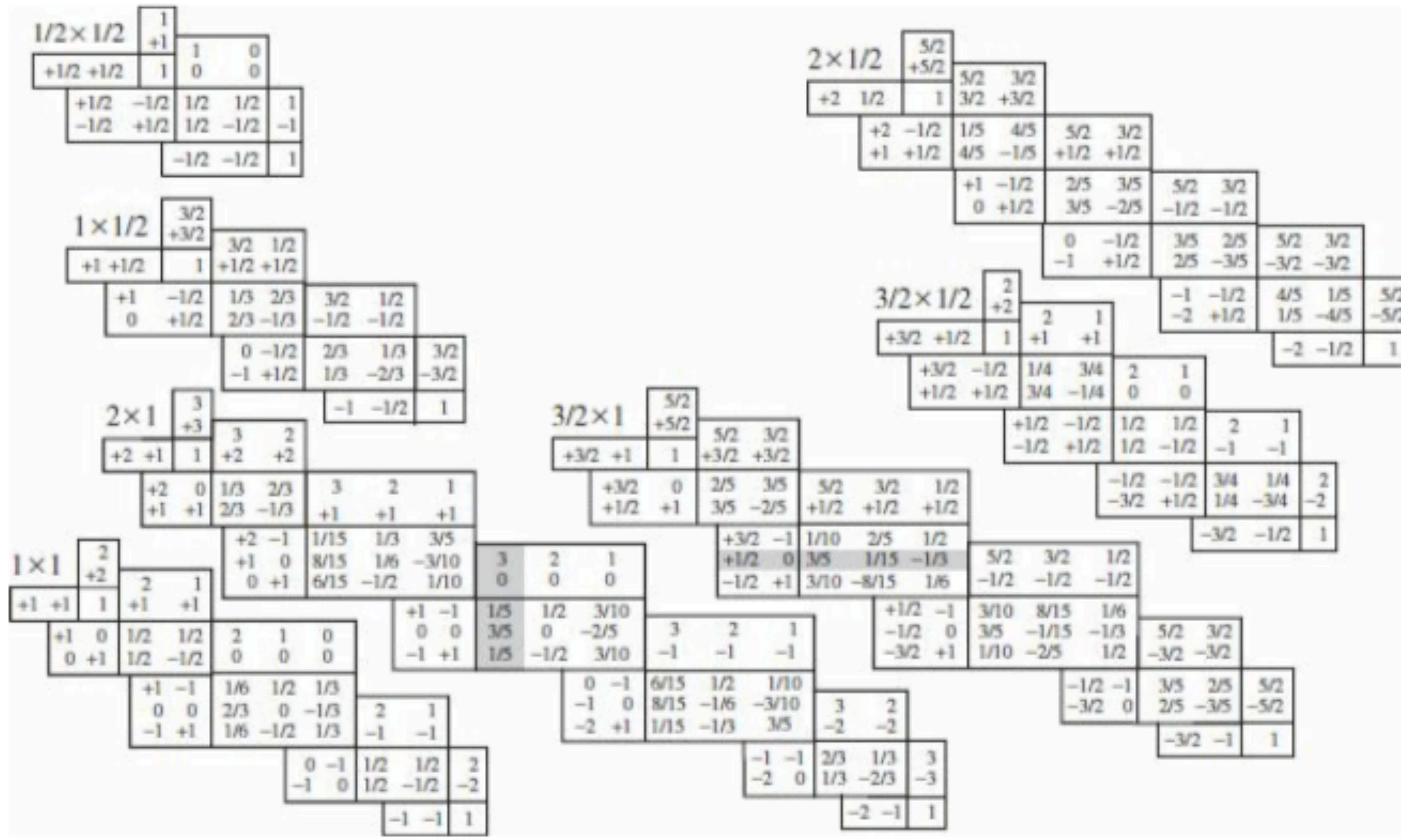

$$|3\ 0\rangle = C_1 |2\ 1\rangle |1\ \ -1\rangle + C_2 |2\ 0\rangle |10\rangle + C_3 |2\ \ -1\rangle |1\ 1\rangle$$

$$= \frac{1}{\sqrt{5}} |2\ 1\rangle |1\ \ -1\rangle + \sqrt{\frac{3}{5}} |2\ 0\rangle |10\rangle + \frac{1}{\sqrt{5}} |2\ \ -1\rangle |1\ 1\rangle$$

II. Two particle wave functions in general

The complete state of an electron puts together both the spatial dependence and the spin of the electron:

$\psi(\vec{r})\chi.$

What happens when we put two particles together?

$\psi(\vec{r}_1, \vec{r}_2)\chi(1,2).$

The symmetrization (or anti-sym.) axiom of quantum mechanics says that it is the *whole* wave function that has a definite symmetry type; e.g., for a fermion

$\psi(\vec{r}_1, \vec{r}_2)\chi(1,2) = -\psi(\vec{r}_2, \vec{r}_1)\chi(2,1).$

This means that we have to consider the full wave function when we are thinking about symmetrization.

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The Pauli exclusion principle states that two fermions (e.g. electrons) cannot occupy the same state.

I can put two electrons in the same spatial wave function as long as I also require that the spin state is the singlet spin state!

We can formalize all of this mathematically. The idea is to introduce a new operator, called the exchange operator and denoted $\hat{P}$. The definition of this operator is that it interchanges two particles

$\hat{P}|(1,2)\rangle = |(2,1)\rangle$.

This operator has a neat property $\hat{P}^2 = 1$, as a matrix this is the unit matrix with ones along the diagonal. This means that the eigenvalues of $\hat{P}$ itself are ± 1. Suppose we had two identical particles…

II. Two particle wave functions in general

This means that the eigenvalues of $\hat{P}$ itself are ± 1. Suppose we had two identical particles…, then the Hamiltonian should treat them exactly the same $m_1 = m_2$ and $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_2, \vec{r}_1)$, but then

$$[\hat{P}, \hat{H}] = 0$$

and hence are compatible observables. From the generalized Ehrenfest result we then have that

$$\frac{d\langle \hat{P} \rangle}{dt} = 0!$$

Any pair of particles that start out in a symmetrized state remain in that state for all time.

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This is also true for n identical particles, they generally satisfy $|(1,2,\ldots,i,\ldots,j,\ldots,n\rangle = \pm|(1,2,\ldots,j,\ldots,i,\ldots,n\rangle$.

III. Atoms

All we'll do today is to write down the Hamiltonian and look at it.:

$$\hat{H} = \sum_{j=1}^{Z}\left\{-\frac{\hbar^2}{2m}\nabla_j^2 - \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Ze^2}{r_j}\right\} + \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\right)\sum_{j\neq k}^{Z}\frac{e^2}{|\vec{r}_j - \vec{r}_k|}$$