Today: Quantum Review Session

I. Last Time
II. Your Questions

I. Last time

* Generalized two-particle states

$\psi(\vec{r}_1, \vec{r}_2)\chi(1,2)$.

*Studied the symmetrization axiom in general

$\psi(\vec{r}_1, \vec{r}_2)\chi(1,2) = \pm\, \psi(\vec{r}_2, \vec{r}_1)\chi(2,1)$,

where anti-symmetry applies to fermions and symmetry to bosons.

*We also introduced an exchange operator $\hat{P}_{12}$, which has the action

$\hat{P}_{12}\,|(1,2)\rangle = |(2,1)\rangle$.

We found that this operator has eigenvalues ± 1. This operator commutes with the Hamiltonian, hence is time-independent, and so if you start in a definite symmetry state you remain in such a state.

I. Two particle wave functions in general

The complete state of an electron puts together both the spatial dependence and the spin of the electron:

$\psi(\vec{r})\chi.$

What happens when we put two particles together?

$\psi(\vec{r}_1, \vec{r}_2)\chi(1,2).$

The symmetrization (or anti-sym.) axiom of quantum mechanics says that it is the *whole* wave function that has a definite symmetry type; e.g., for a fermion

$\psi(\vec{r}_1, \vec{r}_2)\chi(1,2) = -\psi(\vec{r}_2, \vec{r}_1)\chi(2,1).$

This means that we have to consider the full wave function when we are thinking about symmetrization.

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I. Two particle wave functions in general

The Pauli exclusion principle states that two fermions (e.g. electrons) cannot occupy the same state.

I can put two electrons in the same spatial wave function as long as I also require that the spin state is the singlet spin state!

We can formalize all of this mathematically. The idea is to introduce a new operator, called the exchange operator and denoted $\hat{P}$. The definition of this operator is that it interchanges two particles

$\hat{P}|(1,2)\rangle = |(2,1)\rangle$.

This operator has a neat property $\hat{P}^2 = 1$, as a matrix this is the unit matrix with ones along the diagonal. This means that the eigenvalues of $\hat{P}$ itself are ± 1. Suppose we had two identical particles…

I. Two particle wave functions in general

This means that the eigenvalues of $\hat{P}$ itself are ± 1. Suppose we had two identical particles…, then the Hamiltonian should treat them exactly the same $m_1 = m_2$ and $V(\vec{r}_1, \vec{r}_2) = V(\vec{r}_2, \vec{r}_1)$, but then

$[\hat{P}, \hat{H}] = 0$

and hence are compatible observables. From the generalized Ehrenfest result we then have that

$$\frac{d\langle \hat{P} \rangle}{dt} = 0!$$

Any pair of particles that start out in a symmetrized state remain in that state for all time.

The **symmetrization axiom** states that not only do identical particles maintain their symmetrization, but they are required to be in such a state: $|(1,2)\rangle = \pm |(2,1)\rangle$.

I. Two particle wave functions in general

The **symmetrization axiom** states that not only do identical particles maintain their symmetrization, but they are required to be in such a state: $|(1,2)\rangle = \pm |(2,1)\rangle$.

This is also true for n identical particles, they generally satisfy $|(1,2,\ldots,i,\ldots,j,\ldots,n\rangle = \pm |(1,2,\ldots,j,\ldots,i,\ldots,n\rangle$.

I. Atoms

All we'll do today is to write down the Hamiltonian and look at it.:

$$\hat{H} = \sum_{j=1}^{Z} \left\{ -\frac{\hbar^2}{2m}\nabla_j^2 - \left(\frac{1}{4\pi\epsilon_0}\right)\frac{Ze^2}{r_j} \right\} + \frac{1}{2}\left(\frac{1}{4\pi\epsilon_0}\right)\sum_{j\neq k}^{Z} \frac{e^2}{|\vec{r}_j - \vec{r}_k|}$$

II. How can I think of more than one particle wave functions?

The mathematical answer is a tensor product. How can I put two vector spaces together? There's more than one answer.

The first kind of way is called a Cartesian product (physicists usually denote it with $\times$ and mathematicians with $\oplus$. Given two vector spaces V_1 and V_2, with dimensions $\dim V_1 = 1$ and $\dim V_2 = 2$. The Cartesian product just adds these two vector spaces together to get
$V_3 = V_1 \times V_2$,
notice that V_3 has dimension $\dim V_3 = 3$, which is why mathematicians refer to a direct sum. One more way of thinking about this to say that I have added another slot to my vector

$$\begin{pmatrix} * \\ * \end{pmatrix} \to \begin{pmatrix} * \\ * \\ * \end{pmatrix}.$$

II. How can I think of more than one particle wave functions?

The tensor product of two vectors spaces is a new vector space that has dimension equal to the product of the dimensions of the spaces that it is made out of. Given V_2 and V_3 of dimensions 2 and three, their tensor product is

$V = V_2 \otimes V_3$

and is 6 dimensional.

Angular momentum theory

$[L_x, L_y] = i\hbar L_z$.

There are *not* simultaneous eigenfunctions of all three components! We choose to focus on L_z

$L_z Y_\ell^m(\theta, \phi) = m\hbar Y_\ell^m$.

There is a second observable that commutes with L_z, namely L^2,

$L^2 Y_\ell^m = \ell(\ell+1)\hbar^2 Y_\ell^m$.

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The functional form of the Y_ℓ^m was

$Y_\ell^m(\theta, \phi) = N P_\ell^m(\cos\theta) e^{im\phi}$, here $\ell = 0,1,2,3,\ldots$ and

$m = -\ell, -\ell + 1, \ldots, \ell$.

II. G&S Problem 4.47

The Hamiltonian is

$$\hat{H}(r,\theta,\phi) = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2 \text{ (isotropic oscillator).}$$

Working in spherical coordinates

$$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)Y_\ell^m(\theta,\phi).$$

The radial wave equation is

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[\frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u = Eu,$$

where $u = rR(r)$. This is the equation we want to solve. We can attempt a power series solution. First make it homogeneous, next find unites variables to make as clean as possible, finally strip off the asymptotic behavior. This would allow us to write $u(r) = f_{a1}f_{a2}v(r)$.

II. G&S Problem 4.47

$$|\ell s_1 m m_1\rangle = \sum C\,|j m_j\rangle$$

$$\psi_{abc}$$

$$\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)$$