Ethan’s Guest Lecture

I. What are wavefunctions? (a recap)

II. Rethinking operators

III. A time-evolution of a simple two-state system

I. What are wavefunctions? (a recap)

Introduce a general vector $\mathcal{V}(t)$ that lives in Hilbert space.

We showed last week that $\Psi(x,t) = \langle x|\mathcal{V}(t)\rangle$

and that this contains the same information as $\Phi(p,t) = \langle p|\mathcal{V}(t)\rangle$

If you have either Phi or Psi, they can be related by Fourier Transform.

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int \Phi(p,t) e^{ipx/\hbar} dp$$

II. Rethinking operators.

We’ve become used to thinking of operators as prickly functions.

Consider two vectors $|\alpha\rangle = \sum_n a_n|e_n\rangle$ with $a_n = \langle e_n|\alpha\rangle$

and $|\beta\rangle = \sum_n b_n|e_n\rangle$ with $b_n = \langle e_n|\beta\rangle$

related by $|\beta\rangle = \hat{Q}|\alpha\rangle.$

This can be rewritten in the sum notation as $$\sum_n b_n|e_n\rangle = \sum_n a_n\hat{Q}|e_n\rangle$$

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II. Rethinking operators (continued).

Notice we can pick out components by taking the inner product with $|e_m\rangle$

$$\sum_n b_n\langle e_m|e_n\rangle = \sum_n a_n\langle e_m|\hat{Q}|e_n\rangle = \sum_n \langle e_m|\hat{Q}|e_n\rangle a_n$$

What does this result tell us?

$$b_m = \sum_n Q_{mn}a_n$$

The matrix elements tell us how the components transform. The operator is a **matrix.**

III. A time-evolution of a simple two-state system

Consider a system with two linearly independent states: $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then a general vector is $|\mathcal{V}\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

Suppose the Hamiltonian is a matrix given by $\mathcal{H} = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$ with h,g real constants.

If the system starts out in state |1>, what will its state be at time t? $|\mathcal{V}_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|\mathcal{V}_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

III. Determining energy eigenvalues

We'd like to solve the time-dependent Schrodinger equation: $i\hbar\frac{d}{dt}|\mathcal{V}\rangle = \mathcal{H}|\mathcal{V}\rangle$

Start with the time-independent Sch. eq. $\mathcal{H}|v\rangle = E|v\rangle$

This energy-eigenvalue equation can be solved via the roots of the char.eq.

$$\det(\mathcal{H} - E\mathbb{1}) = 0 = \det\begin{pmatrix} h - E & g \\ g & h - E \end{pmatrix} = (h-E)^2 - g^2$$

With roots (eigenvalues) found, like so: $h - E = \pm g \Longrightarrow E_\pm = h \pm g$

$$|\mathcal{V}_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$E_{\pm} = h \pm g$$

III. Determining eigenvectors

Now to solve for the eigenvectors, rewriting $\mathcal{H}|v\rangle = E|v\rangle$ as $\begin{pmatrix} h & g \\ g & h \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (h \pm g) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

We have $\begin{pmatrix} h\alpha + g\beta \\ g\alpha + h\beta \end{pmatrix} = \begin{pmatrix} (h \pm g)\alpha \\ (h \pm g)\beta \end{pmatrix} \implies h\alpha + g\beta = (h \pm g)\alpha \implies \beta = \pm\alpha$

Evidently, our normalized eigenvectors are $|v_{\pm}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$

$$|\mathcal{V}_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$E_\pm = h \pm g$$

III. Determining the full state vector.

$$|v_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

Expanding the initial vector, $|\mathcal{V}_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|v_+\rangle + |v_-\rangle).$

And adding a wiggle factor, $|\mathcal{V}(t)\rangle = \frac{1}{\sqrt{2}}[e^{-i(h+g)t/\hbar}|v_+\rangle + e^{-i(h-g)t/\hbar}|v_-\rangle]$

$$= \frac{1}{2}e^{-iht/\hbar}\left[e^{-igt/\hbar}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{igt/\hbar}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{2}e^{-iht/\hbar}\begin{pmatrix} e^{-igt/\hbar} + e^{igt/\hbar} \\ e^{-igt/\hbar} - e^{igt/\hbar} \end{pmatrix} = e^{-iht/\hbar}\begin{pmatrix} \cos(\frac{gt}{\hbar}) \\ -i\sin(\frac{gt}{\hbar}) \end{pmatrix}$$

Check: at t=0, $e^0 \begin{pmatrix} \cos(0) \\ -i\sin(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\mathcal{V}_0\rangle$

Appendix: Normalizing the eigenvectors

First, we find the length of the eigenvectors $\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$

$$1^2 + (\pm 1)^2 = 2 = ||v_+\rangle|^2$$

Dividing the eigenvectors by their length will give the normalized eigenvectors

$$\frac{1}{||v_+\rangle|} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

Check: $(\frac{1}{\sqrt{2}})^2 + (\pm\frac{1}{\sqrt{2}})^2 = \frac{1}{2} + \frac{1}{2} = 1$