Ethan's Guest Lecture

- I. What are wavefunctions? (a recap)
- II. Rethinking operators
- III. A time-evolution of a simple two-state system

I. What are wavefunctions? (a recap)

Introduce a general vector $\mathcal{V}(t)$ that lives in Hilbert space.

We showed last week that $\Psi(x,t) = \langle x | \mathcal{V}(t) \rangle$

and that this contains the same information as $\Phi(p,t) = \langle p | \mathcal{V}(t) \rangle$

If you have either Phi or Psi, they can be related by Fourier Transform.

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int \Phi(p,t) e^{ipx/\hbar} dp$$

II. Rethinking operators.

We've become used to thinking of operators as prickly functions.

Consider two vectors
$$|\alpha\rangle = \sum_{n} a_{n} |e_{n}\rangle$$
 with $a_{n} = \langle e_{n} | \alpha \rangle$
and $|\beta\rangle = \sum_{n} b_{n} |e_{n}\rangle$ with $b_{n} = \langle e_{n} | \beta \rangle$

related by $|\beta\rangle = \hat{Q}|\alpha\rangle$.

This can be rewritten in the sum notation as

$$\sum_{n} b_n |e_n\rangle = \sum_{n} a_n \hat{Q} |e_n\rangle$$

$\sum_{n} b_{n} |e_{n}\rangle = \sum_{n} a_{n} \hat{Q} |e_{n}\rangle$

II. Rethinking operators (continued).

Notice we can pick out components by taking the inner product with $|e_m\rangle$

$$\sum_{n} b_n \langle e_m | e_n \rangle = \sum_{n} a_n \langle e_m | \hat{Q} | e_n \rangle = \sum_{n} \langle e_m | \hat{Q} | e_n \rangle a_n$$

What does this result tell us?

$$b_m = \sum_n Q_{mn} a_n$$

The matrix elements tell us how the components transform. The operator is a matrix.

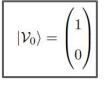
III. A time-evolution of a simple two-state system

Consider a system with two linearly independent states: $|1\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} |2\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Then a general vector is
$$|\mathcal{V}\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

Suppose the Hamiltonian is a matrix given by $\mathcal{H} = \begin{pmatrix} h & g \\ g & h \end{pmatrix}$ with h,g real constants.

If the system starts out in state |1>, what will its state be at time t? $|\mathcal{V}_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



III. Determining energy eigenvalues

We'd like to solve the time-dependent Schrodinger equation:

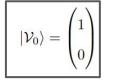
$$i\hbarrac{d}{dt}|\mathcal{V}
angle=\mathcal{H}|\mathcal{V}
angle$$

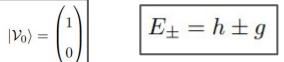
Start with the time-independent Sch. eq. $\mathcal{H}|v\rangle = E|v\rangle$

This energy-eigenvalue equation can be solved via the roots of the char.eq.

$$\det(\mathcal{H} - E\mathbb{1}) = 0 = \det\begin{pmatrix} h - E & g\\ g & h - E \end{pmatrix} = (h - E)^2 - g^2$$

With roots (eigenvalues) found, like so: $h - E = \pm g$ \longrightarrow $E_{\pm} = h \pm g$





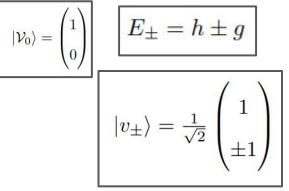
III. Determining eigenvectors

Now to solve for the eigenvectors, rewriting $\mathcal{H}|v\rangle = E|v\rangle$ as $\begin{pmatrix} h & g \\ g & h \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (h \pm g) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

We have
$$\begin{pmatrix} h\alpha + g\beta \\ g\alpha + h\beta \end{pmatrix} = \begin{pmatrix} (h \pm g)\alpha \\ (h \pm g)\beta \end{pmatrix} \implies h\alpha + g\beta = (h \pm g)\alpha \implies \beta = \pm \alpha$$

Evidently, our normalized eigenvectors are

$$v_{\pm}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\ \pm 1 \end{pmatrix}$$



III. Determining the full state vector.

Expanding the initial vector,
$$|\mathcal{V}_0\rangle = \begin{pmatrix} 1\\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|v_+\rangle + |v_-\rangle).$$

And adding a wiggle factor, $|\mathcal{V}(t)\rangle = \frac{1}{\sqrt{2}} [e^{-i(h+g)t/\hbar} |v_+\rangle + e^{-i(h-g)t/\hbar} |v_-\rangle]$

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$$= \frac{1}{2}e^{-i\hbar t/\hbar} \left[e^{-igt/\hbar} \begin{pmatrix} 1\\1 \end{pmatrix} + e^{igt/\hbar} \begin{pmatrix} 1\\-1 \end{pmatrix} \right] = \frac{1}{2}e^{-i\hbar t/\hbar} \begin{pmatrix} e^{-igt/\hbar} + e^{igt/\hbar}\\e^{-igt/\hbar} - e^{igt/\hbar} \end{pmatrix} \right] = e^{-i\hbar t/\hbar} \begin{pmatrix} \cos(\frac{gt}{\hbar})\\-i\sin(\frac{gt}{\hbar}) \end{pmatrix}$$

Check: at t=0, $e^{0} \begin{pmatrix} \cos(0)\\-i\sin(0) \end{pmatrix} = \begin{pmatrix} 1\\0 \end{pmatrix} = |\mathcal{V}_{0}\rangle$

Appendix: Normalizing the eigenvectors

First, we find the length of the eigenvectors $\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$

$$1^2 + (\pm 1)^2 = 2 = ||v_+\rangle|^2$$

Dividing the eigenvectors by their length will give the normalized eigenvectors

$$\frac{1}{||v_{+}\rangle|} \begin{pmatrix} 1\\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\ \pm 1 \end{pmatrix}$$

Check:
$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\pm\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$