Homework 10

Due Friday, April 22nd at 5pm

Read Chapter 20 of Hartle's *Gravity*.

1. In the Schwarzschild geometry consider the following function:

$$f(x^\alpha) = \frac{5t^2 - 2r^2}{(2M)^2}$$

where t and r are the usual Schwarzschild coordinates. Find the coordinate basis components $(\nabla f)^\alpha$ of the gradient of f.

2. Hartle's Eq. (20.81) gives the upstairs coordinate basis components of a set of four vectors $\{\mathbf{e}_{\hat{\alpha}}\}$ constituting an orthonormal frame in the Schwarzschild geometry.

   (a) Verify explicitly that this is an orthonormal set of vectors.

   (b) Find the downstairs coordinate basis components of each of these vectors.

   (c) Find the upstairs coordinate basis components of the basis ea that is dual to the given set of basis vectors.

   (d) Consider a vector $\mathbf{a}$ with upstairs coordinate basis components

$$a^\alpha = (4, 3, 0, 0)$$

   at the point $(0, 3M, 0, 0)$. Find the components $a^{\hat{\alpha}}$ and $a_{\hat{\alpha}}$ of this vector in the given orthonormal frame.

3. Given the map notation for a tensor R is $R : V^* \times V \times V \times V \to \mathbb{R}$:

   (a) What is the rank of this tensor? Display the components of the tensor R in index notation?

   (b) Write out $R(\boldsymbol{\omega}, \mathbf{a}, \mathbf{b}, \mathbf{c})$ in component notation.

4. Work out the expression for the covariant derivative $\nabla_\gamma t_{\alpha\beta}$ analogous to Hartle's Eqs. (20.65) and (20.68).

5. In Hartle's Example 20.8 the acceleration four-vector of a stationary observer in the Schwarzschild geometry could have been computed using

$$a^\alpha = u^\beta \nabla_\beta u^\alpha$$

and Hartle's formula (20.67). Show that the same result, Hartle's (20.61), could have been obtained this way.