Homework 11

Due Friday, May 6th at 5pm

Read Chapter 21 of Hartle's *Gravity*.

1. *Null Geodesics with Nonaffine Parametrization* As we showed in class (see also Hartle's section 8.3), when the tangent vector to a null geodesic $\mathbf{u}$ is parametrized with an affine parameter λ, it obeys the geodesic equation
$$\nabla_{\mathbf{u}}\mathbf{u} = 0.$$
Show that even if a nonaffine parameter is used,
$$\nabla_{\mathbf{u}}\mathbf{u} = -\kappa\mathbf{u}$$
for some function κ of the parameter λ.

2. *Suiface Gravity of a Black Hole* In the geometry of a spherical black hole, the Killing vector $\boldsymbol{\xi} = \partial/\partial t$ corresponding to time translation invariance is tangent to the null geodesics that generate the horizon. From the last problem you know that this means
$$\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi} = -\kappa\boldsymbol{\xi}$$
for a constant of proportionally κ, which is called the surface gravity of the black hole. Evaluate this relation to find the value of κ for a Schwarzschild black hole in terms of its mass, M. Be sure to use a coordinate system that is nonsingular on the horizon such as the Eddington-Finkelstein coordinates discussed in class (Hartle's Section 12.1). If you attended the guest lecture that I gave in Paul's class then you may find it interesting that the temperature of a black hole is really derived in terms of this surface gravity. So, this quantity plays a central role in black hole thermodynamics.

3. *Killing's equation* In class (see also Hartle's section 8.2) a Killing vector corresponding to a symmetry of a metric was defined in a coordinate system in which the metric was independent of one coordinate, x^1. The components of the corresponding Killing vector $\boldsymbol{\xi}$ are then
$$\xi^\alpha = (0, 1, 0, 0).$$
By explicit calculation show that
$$\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 0.$$
This is Killing's equation. It is a general characterization of Killing vectors in the sense that any solution corresponds to a symmetry of the metric.

4. *Physical approach to curvature* The curvature formula that we derived in class assumed that the curve was parametrized by arc length. This is particularly elegant, but, unfortunately, it can be difficult to find the arc length analytically for many curves. So, it is useful to derive

a formula for the curvature using any parametrization. We will take a physical approach to this derivation.

The acceleration can be decomposed into a part that is along the curve, the longitudinal acceleration, and a part that is perpendicular to the curve. The latter part, which physicists call the centripetal acceleration, is closely related to the curvature. We have

$$\vec{a} = a_c \hat{n} + a_l \hat{t},$$

where a_c is the centripetal acceleration, a_l is the longitudinal acceleration, $\hat{n}$ is the unit vector normal to the curve, and $\hat{t}$ is the unit tangent vector to the curve.

(a) Suppose you are given a curve $(x(t), y(t))$ parametrized by time. Find the normalized tangent $\hat{t}$ to this curve.

(b) Using the expression from (a) find the unit normal vector to the curve, $\hat{n}$.

(c) From introductory physics you may know that for a circular trajectory $a_c = v^2/R$ where v is the (constant) speed along the curve and R is the radius of the circle. As we discussed in class, the osculating circle gives you a realization of this acceleration at every point of the trajectory and hence

$$\vec{a} = v^2 \kappa \hat{n} + a_l \hat{t},$$

where $\kappa = 1/R$ is the curvature of the curve at that point. Find a more general formula for the curvature of your curve with general parametrization $(x(t), y(t))$ by first calculating $\vec{a}$ and then dotting it with your expression for $\hat{n}$ from the last part and solving for κ. [Hint: You can derive the acceleration of the curve $\vec{a}$ directly from its definition as a second derivative.]

(d) Use your new formula from (c) to derive the curvature of a parabola $y = x^2$ and that of an a ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

at a general point along these curves.

5. In class we claimed that $\Gamma^\alpha_{\beta\gamma}$ was symmetric in its lower two indices. Let's prove this. Hartle's equation (20.48) gives the expression for the components of the second-rank tensor that results from covariant differentiation in a local inertial frame where all the $\Gamma^\alpha_{\beta\gamma}$'s vanish. Use the transformation law for tensors, Hartle's (20.45), to obtain an expression for the $\Gamma^\alpha_{\beta\gamma}$'s in a general coordinate system. Use this result to show that $\Gamma^\alpha_{\beta\gamma}$ is symmetric in β and γ.