Homework 12

Due Friday, May 13th at 5pm

Read Chapters 16 & 23 of Hartle's *Gravity*.

1. This week you derived the Schwarzschild solution from scratch, a remarkable accomplishment. However, in the process I gave you a worksheet to start from and you really need to convince yourself that the sheet I gave you is correct. In this problem you will do a few cases by hand—an important thing to have done at least once. Given a metric of the form

$$ds^2 = -A(dx^0)^2 + B(dx^1)^2 + C(dx^2)^2 + D(dx^3)^2,$$

where A, B, C, and D in general depend on all four coordinates. Calculate Γ^0_{11} and R_{12} by hand and check them against the worksheet (which you can download here). Feel free to use the other given $\Gamma^\alpha_{\beta\gamma}$ on the worksheet in your calculation of R_{12}.

2. After having solved 1. above, you will probably not be anxious to do more of these calculations by hand. It's useful to be able to do these calculations in a symbolic manipulation program as well. Leonard Parker, a collaborator of Prof. Hartle's book team, has created a *Mathematica* notebook to do exactly this, which you can download here. Download the "Curvature and the Einstein Equation" notebook at the bottom of the page.

   Use this notebook to calculate all of the Christoffel symbols and all of the components of the Ricci tensor for the metric from 1. Note you will have to make several notational adjustments and learn to interpret the output of the program properly; this is typical of using a computer program. [For example, you won't want to use the letter capital D in Mathematica where it is used for other purposes.] Confirm that all the Christoffel results agree with the worksheet. Don't print out your results, they will be too long. Instead, rewrite R_{11} in the notation used on the worksheet and confirm that the worksheet has the correct result. This transcribed result is what you should turn in.

3. Show that the gravitational wave spacetime in Hartle's Eq. (16.2) has three Killing vectors: $(0, 1, 0, 0), (0, 0, 1, 0)$, and $(1, 0, 0, 1)$.

4. The equation for an ellipse is $x^2/a^2 + y^2/b^2 = 1$, where a is the semimajor axis and b is the semiminor axis if $a > b$. Show that an initial circle of test particles distorts into an ellipse according to Hartle's Eq. (16.13) to lowest order in a and compute the semimajor and semiminor axes as a function of time.

5. *Deriving the Static Weak Field Metric* In this problem you will derive the static, weak-field metric, Hartle's Eq. (21.25), using the Einstein equation. This is the most general solution to the linearized, vacuum Einstein equation. These efforts will also allow you to identify the physical interpretation of the arbitrary constant in your Schwarzschild solution to the Einstein equation.

   (**a**) Argue that the metric perturbations $h_{\alpha\beta}$ for a time-independent source should be unchanged by $t \to -t$ and that this means $h_{it} = h_{ti} = 0$.

(**b**) Show that the residual gauge freedom analogous to that discussed in the subsection "More Gauge" can be used to make h_{ij} diagonal without affecting either $h_{it} = 0$ or the Lorentz gauge condition.

(**c**) Show that then (21.25) is the unique asymptotically flat solution of the equations of linearized gravity. Feel free to use the diagonal metric worksheet if it is of value to you.

(**d**) You know the solution for the weak-field metric outside a spherical body is $\Phi = GM/r$. In the limit that $r \gg 2GM$ the Schwarzschild metric should reduce to the weak field metric. Use this limit to identify the constant R_S in your previous solution of the Einstein equation.