Homework 3

Due Friday, February 19th at 5pm

Read Chapters 4 and 5 of Hartle's *Gravity*. Begin reading chapter 6 for next week.

1. In class we discussed how different the Lorentzian geometry of the x-ct plane is from that of the x-y plane. Let us elaborate on this a little.

   (a) Draw all points that are a constant *distance*, of say 1 meter, from the origin of the x-y plane. What do we call this curve?

   (b) Draw all points that are a constant *spacetime interval*, of say $\Delta s^2 = -1$ meter^2, from the origin of the x-ct plane. What do we call this curve? What are its asymptotes? Are these points timelike, null, or spacelike separated from the origin?

   (c) Draw all points that are a constant *spacetime interval*, of say $\Delta s^2 = 1$ meter^2, from the origin of the x-ct plane. Are these points timelike, null, or spacelike separated from the origin?

   (d) Draw a right triangle, with one vertex at the origin, in the x-ct plane whose hypotenuse is timelike. Denote the angle between the ct axis and the hypotenuse by θ. If the edges of the triangle represent distances of $c\Delta t$ and Δx respectively, what is the slope of the hypotenuse? Calculate this slope a second time using hyperbolic trigonometry. Setting these two expressions equal, what do you find to be the relationship between the boost angle θ and the velocity of a particle moving along the hypotenuse?

2. Show that for two timelike separated events, there is some inertial frame in which $\Delta t \neq 0, \vec{\Delta x} = 0$ [that is, $(\Delta x, \Delta y, \Delta z) = (0, 0, 0)$]. Show that for two spacelike separated events there is an inertial frame where $\Delta t = 0, \vec{\Delta x} \neq 0$.

3. The goal of this problem is to find the geodesics on a cone. Consider a particle confined to move on the surface of a circular cone with its axis on the z-axis, vertex at the origin (pointing down), and half-angle α. The particle's position can be specified by two coordinates, which you can choose to be the coordinates (ρ, ϕ) of cylindrical polar coordinates.

   (a) Write down the equations that give the three Cartesian coordinates of the particle in terms of the generalized coordinates (ρ, ϕ) and use these to derive the line element on the surface of the cone. Simplify your expression as much as possible.

   (b) Find the Euler-Lagrange equations that guarantee the extremization of the arc length on the cone. You will probably want to choose ρ as your independent variable so that $\phi = \phi(\rho)$.

   (c) Simplify your Euler-Lagrange equations as much as possible and separate variables so that you can express ϕ as an integral over ρ. If you put this integral into *Mathematica* it is likely to spit out an ugly complex answer. You should have an integral that looks like it could be done by trigonometric substitution, but when you try it doesn't work. Instead try making a hyperbolic trigonometric substitution, this should work quite nicely. [If you are feeling ambitious, use *Mathematica* to plot the cone and your geodesics on its surface. This is not required.]

4. Consider twins, Stella and Terrance. Stella goes off in a straight line traveling at a speed of $\sqrt{\frac{3}{4}}c$ for 5 years as measured on *her* clock then reverses and returns at the same speed. Terrance remains at home on earth. Make a spacetime diagram showing the motion of Stella and Terrance from Terrance's point of view. When they return, what is the difference in ages between Stella and Terrance? (This is quite similar to Problem 5 from last week.)

5. Same setup as the last problem. It can be confusing that the situation between the two twins is not symmetrical. To address this issue consider the following method for analyzing the problem: Consider the problem from Stella's perspective, she sees Terrance head off in the opposite direction at the same speed.

   (a) Thus during the outgoing trip how much does she calculate him as aging?

   (b) Same question for the incoming trip?

   This indicates that she thinks that Terrance is younger at the end of the trip. However, it neglects the fact that she was the one who had to do the turning around!

   (c) Make another copy of your spacetime diagram from Problem 4. On this spacetime diagram draw the lines of constant time for Stella's outgoing trip.

   (d) Using a different color (or dashing your lines) draw the lines of constant time for Stella's incoming trip.

   (e) At the turnaround event, do these two sets of lines agree?

   (f) Explain how your answer to e) resolves the fact that Stella is truly the younger twin upon returning to earth.