# Homework 4

Due Friday, February 26th at 5pm

Read Chapters 5 and 6 of Hartle's *Gravity*.

1. Consider two four-vectors $\mathbf{a}$ and $\mathbf{b}$ whose components are given by

$$\mathbf{a} = (-2, 0, 1, 1),$$
$$\mathbf{b} = (5, 0, 3, 4).$$

   (a) Is $\mathbf{a}$ timelike, spacelike, or null? Is $\mathbf{b}$ timelike, spacelike, or null?

   (b) Compute $\mathbf{a} - 5\mathbf{b}$.

   (c) Compute $\mathbf{a} \cdot \mathbf{b}$.

   (d) The scalar product between two three-vectors can be written as

$$\vec{c} \cdot \vec{d} = cd \cos\theta_{cd}$$

   where $c$ and $d$ are the lengths of $\vec{c}$ and $\vec{b}$, respectively, and $\theta_{cd}$ is the angle between them. Show that an analogous formula holds for two arbitrary timelike four-vectors $\mathbf{c}$ and $\mathbf{d}$,

$$\mathbf{c} \cdot \mathbf{d} = -c\ d \cosh\theta_{cd},$$

   where $c = (-\mathbf{c} \cdot \mathbf{c})^{1/2}$, $d = (-\mathbf{d} \cdot \mathbf{d})^{1/2}$, and $\theta_{cd}$ is the parameter defined in Hartle's equation (4.18) that describes the Lorentz boost between the frame where an observer whose world line points along $\mathbf{c}$ is at rest and the frame where an observer whose world line points along $\mathbf{d}$ is at rest.

2. A free particle is moving along the $x$-axis of an inertial frame with speed $\frac{dx}{dt} = V$ passing through the origin at $t = 0$. Express the particles's world line parametrically in terms of $V$ using the proper time $\tau$ as the parameter.

3. Work out the components of the four-acceleration vector $\mathbf{a} = d\mathbf{u}/d\tau$ in terms of the three-velocity $\vec{V}$ and the three-acceleration $\vec{a} = dV/dt$ to obtain an expression analogous to Hartle's equation (5.28). Using this expression and Eq. (5.28), verify explicitly that $\mathbf{a} \cdot \mathbf{u} = 0$.

4. We have been building up to a fun calculation: the goal of this problem is to find the geodesics on a paraboloid. You may have noticed that all of our geodesic calculations up to now have been straight lines in disguise. If you had wanted to you would have been able to find these geodesics by drawing straight lines on a piece of paper and then cutting it up and pasting it together in a variety of ways. In this problem you will do your first calculation of geodesics on a genuinely curved surface that is not so easy to construct out of a piece of paper, the paraboloid.

   Consider a particle confined to move on the surface of a paraboloid, which is given by the equation $z = x^2 + y^2$. Again we will find it convenient to work in cylindrical coordinates

$(\rho, \phi, z)$. Find the equation of the paraboloid in these coordinates. Because the surface is two-dimensional, the particle's position can be specified by two coordinates, which you can choose to be the two coordinates $(\rho, \phi)$.

(a) Write down the equations that give the three Cartesian coordinates of the particle in terms of the generalized coordinates $(\rho, \phi)$ and use these to derive the line element on the paraboloid.

(b) Find the Euler-Lagrange equations that guarantee the extremization of the arc length on the paraboloid. You will probably want to choose $\rho$ as your independent variable so that $\phi = \phi(\rho)$.

(c) Simplify your Euler-Lagrange equations as much as possible and separate variables so that you can express $\phi$ as an integral over $\rho$. If you put this integral into *Mathematica* it is likely to spit out an answer in terms of a function you've never met. However, you should be able to simplify it by trigonometric substitution (either regular trig functions or hyperbolic). After the substitution you can ask *Mathematica* again and you should get a closed form answer in terms of functions that you know, although it is not simple. That's ok, it is neat that you can solve this analytically at all. [If you are feeling ambitious, use *Mathematica* to plot the paraboloid and your geodesics on its surface. This is not required.]

5. A particle is moving along the $x$-axis. It is uniformly accelerated in the sense that the acceleration measured in its instantaneous rest frame is always $g$, a constant. Find $x$ and $t$ as functions of the proper time $\tau$ assuming the particle passes through $x_o$ at time $t = 0$ with zero velocity. Draw the world line of the particle on a spacetime diagram.