Homework 6

Due Friday, March 11th at 5pm

Read Chapter 7 of Hartle's *Gravity.*

1. Three observers are standing near each other on the surface of the Earth. Each holds an accurate atomic clock. At time $t = 0$ all the clocks are synchronized. At $t = 0$ the first observer throws his clock straight up so that it returns at time T as measured by the clock of the second observer, who holds her clock in her hand for the entire time interval. The third observer carries his clock up to the maximum height the thrown clock reaches and back down, moving with constant speed on each leg of the trip and returning in time T.

   Calculate the total elapsed time measured on each clock assuming that the maximum height is much smaller than the radius of the Earth. Include gravitational effects but calculate to order $1/c^2$ only using nonrelativistic trajectories. Which clock registers the *longest* time? Why is this?

2. (a) Building on our explorations of accelerated observers, transform the line element of special relativity from the usual (t, x, y, z) rectangular coordinates to new coordinates (t', x', y', z') related by

$$t = \left(\frac{c}{g} + \frac{x'}{c}\right) \sinh\left(\frac{gt'}{c}\right)$$

$$x = c\left(\frac{c}{g} + \frac{x'}{c}\right) \cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g}$$

$$y = y', \qquad z = z'.$$

   for a constant g with the dimensions of acceleration.
   (b) For $gt'/c \ll 1$, show that this corresponds to a transformation to a uniform!y accelerated frame in Newtonian mechanics.
   (c) Show that a clock at rest in this frame at $x' = h$ runs fast compared to a clock at rest at $x' = 0$ by a factor $(1 + gh/c^2)$. How is this related to the equivalence principle idea?

3. (a) An accelerated laboratory has a bottom at $x' = 0$ and a top at $x' = h$, both with extent in the y'- and z'-direction. Use the line element derived in part (a) of the last problem to show that the height of the laboratory remains constant in time, i.e., the laboratory moves rigidly.
   (b) Compute the invariant acceleration $a = (\mathbf{a} \cdot \mathbf{a})^{1/2}$, where $a^\alpha = d^2x/d\tau^2$, and show that it is different for the top and bottom of the laboratory. Explain why we use a plus sign under the square root in the definition of the magnitude of the acceleration.

4. A GPS satellite emits signals at a constant rate as measured by an onboard clock. Calculate the fractional difference in the rate at which these are received by an identical clock on the surface of the Earth. Take both the effects of special relativity and gravitation into account to leading order in $1/c^2$. For simplicity assume the satellite is in a circular equatorial orbit, the ground-based clock is on the equator, and that the angle between the propagation of the signal and the velocity of the satellite is 90° in the instantaneous rest frame of the receiver.

5. Consider a particle moving in a circular orbit about the Earth of radius R. Suppose the geometry of spacetime outside the Earth is given by the static weak field metric with Φ the gravitational potential of the Earth, $\Phi = -GM_{\oplus}/r$. Let P be the period of the orbit measured in the coordinate time t of the weak field metric. Consider two events A and B located at the same spatial position on the orbit but separated in t by the period P. The particle's world line is a curve of extremal proper time between A and B. Analyze the question of whether the world line is a curve of longest, shortest, or just extremal proper time by calculating the proper time to first order in $1/c^2$ along the following curves between A and B:

   (a) The orbit of the particle itself.

   (b) The world line of an observer who remains fixed in space between A and B.

   (c) The world line of a photon that moves radially away from A and reverses direction in time to return to B in a time P.

   (d) Can you find another curve of extremal proper time that connects A and B?