Homework 7

Due Friday, April 1st at 5pm

Read Chapters 8 & 9 of Hartle's *Gravity*.

1. (**a**) In the singular line element for the plane that we derived in class
$$dS^2 = \frac{a^4}{r'^4}(dr'^2 + r'^2 d\phi^2),$$
show that the distance between $r' = 0$ and a point with any finite value of r' is infinite.
(**b**) Explain why this is true even though this metric is just a coordinate transformation of the usual flat plane.
(**c**) Find the distance between $r' = 5$ and $r' = \infty$ along the line $\phi = 0$.

2. The following line element corresponds to flat spacetime:
$$ds^2 = -dt^2 + 2dxdt + dy^2 + dz^2.$$
Find a coordinate transformation that puts the line element in the usual flat spacetime form.

3. Consider the two-dimensional spacetime spanned by coordinates (v, x) with the line element
$$ds^2 = -xdv^2 + 2dvdx.$$
(**a**) Calculate the light cone at a point (v, x). We have not done this in class yet, but you know all the ideas. Recall that light rays are null, so that $ds^2 = 0$. Use this to derive a differential equation from the metric and then solve it to find the light cones.
(**b**) Draw a (v, x) spacetime diagram showing how the light cones change with x.
(**c**) Show that a particle can cross from positive x to negative x but cannot cross from negative x to positive x.
(*Comment*: The light cone structure of this model spacetime is in many ways analogous to that of the black-hole spacetimes we will be studying shorty, in particular in having a surface such as $x = 0$, out from which you cannot get.)

4. *Transformation Law for the Metric* A general coordinate transformation is specified by four functions $x'^\alpha = x'^\alpha(x^\beta)$.
(**a**) Use the chain rule to express the differentials of the old and new coordinates in terms of one another by
$$dx^\alpha = \frac{\partial x^\alpha}{\partial x'^\gamma} dx'^\gamma.$$
(**b**) Substitute this into the general line element $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$ to express the transformed metric $g'_{\delta\gamma}$ in terms of $g_{\alpha\beta}$ and $\partial x^\alpha / \partial x'^\gamma$ etc. Make sure your answer is consistent with the summation convention.

5. The explanation of this problem is long, but what you have to do is of a comparable length to other problems.

(**a**) Use the mathematical fact that any real symmetric matrix can be diagonalized by an orthogonal matrix to show that any metric can be diagonalized at one point P by a linear transformation of the form

$$x'^{\alpha} = M^{\alpha}_{\beta} x^{\beta}.$$

In particular, make clear the connection between orthogonal matrix of the theorem and $g_{\alpha\beta}(x_P)$, and between M^{α}_{β} and the components of the orthogonal diagonalizing matrix.

(**b**) The argument in Section 7.4 of Hartle's book shows that at a point P there are coordinates in which the value of the metric takes its flat space form $\eta_{\alpha\beta}$. But we have been arguing that there are coordinates in which the first derivatives of the metric vanish at P, as they do in flat space? What about the second derivatives? The following counting argument, although not conclusive, shows how far one can go.

You worked out the rule for transforming the metric between one coordinate system and another above. This can be expanded as a power (Taylor) series about x_P:

$$\begin{aligned} x^{\alpha}(x'^{\beta}) = x^{\alpha}(x'^{\beta}_P) &+ \left(\frac{\partial x^{\alpha}}{\partial x'^{\beta}}\right)_{x_P} (x'^{\beta} - x'^{\beta}_P) \\ &+ \frac{1}{2}\left(\frac{\partial^2 x^{\alpha}}{\partial x'^{\beta}\partial x'^{\gamma}}\right)_{x_P} (x'^{\beta} - x'^{\beta}_P)(x'^{\gamma} - x'^{\gamma}_P) \\ &+ \frac{1}{6}\left(\frac{\partial^3 x^{\alpha}}{\partial x'^{\beta} x'^{\gamma} x'^{\delta}}\right)_{x_P} (x'^{\beta} - x'^{\beta}_P)(x'^{\gamma} - x'^{\gamma}_P)(x'^{\delta} - x'^{\delta}_P) + \cdots . \end{aligned}$$

At the point x^{α}_P there are 16 numbers $(\partial x^{\alpha}/\partial x'^{\beta})_{x_P}$ to adjust to make the transformed values of the metric $g'_{\alpha\beta}$ equal to $\eta_{\alpha\beta}$. Since there are only 10 components of $g'_{\alpha\beta}$, we can do this and still have 6 numbers to spare! These 6 degrees of freedom correspond exactly to the 3 rotations and 3 Lorentz boosts, which leave $\eta_{\alpha\beta}$ unchanged. Following this line of reasoning, fill in the rest of the spaces in the following table to show that there is enough freedom in coordinate transformations to make the first derivatives of the metric vanish in addition to setting $g'_{\alpha\beta}(x'_P) = \eta_{\alpha\beta}$ but not the second derivatives:

	Conditions	Numbers
$g'_{\alpha\beta} = \eta_{\alpha\beta}$	10	16
$\partial g'_{\alpha\beta}/\partial x'^{\gamma}$	?	?
$\partial^2 g'_{\alpha\beta}/\partial x'^{\gamma}\partial x'^{\delta} = 0$	?	?

When properly organized, the second derivatives that cannot be transformed away are the measure of spacetime curvature, as we shall see later in the course. How many of them are there?