Homework 8

Due Friday, April 8th at 5pm

Read Chapter 12 of Hartle's *Gravity*.

1. In usual spherical coordinates the metric on a two-dimensional sphere is

$$dS^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where a is the constant radius of the sphere.

(**a**) Calculate the Christoffel symbols "by hand", that is, using the explicit formula in terms of the metric.

(**b**) Show that a great circle is a solution of the geodesic equations that we derived before the break. [Hint: Make use of the freedom to orient the coordinates so the equation of a great circle is simple.]

2. An observer falls radially inward toward a black hole of mass M whose exterior geometry is the Schwarzschild geometry, starting with zero kinetic energy at infinity. How much time does it take, as measured on the observer's clock, to pass between the radii $6M$ and $2M$?

3. Two particles fall radially in from infinity in the Schwarzschild geometry. One starts with $e = 1$, the other with $e = 2$. A stationary observer at $r = 6M$ measures the speed of each when they pass by. How much faster is the second particle moving at that point?

4. A spaceship is moving without power in a circular orbit about a black hole of mass M. (The exterior geometry is the Schwarzschild geometry.) The Schwarzschild radius of the orbit is $7M$. (**a**) What is the period of the orbit as measured by an observer at infinity? (**b**) What is the period of the orbit as measured by a clock in the spaceship?

5. *Precession oft he Perihelion of a Planet* To find the first order in $1/c^2$ relativistic correction to the angle $\Delta\phi$ swept out in one bound orbit, one might be tempted to expand the integrand in Hartle's Eq. (9.52) in the small quantity $2GM\ell^2/c^2r^3$ and keep only the first two terms. This would be a mistake because the resulting integral would diverge near a turning point such $\int^{r_2} dr/(r_2 - r)^{3/2}$, whereas the original integral is finite. There are several ways of rewriting the integrand so it can be expanded. One trick is to factor $(1 - 2GM/c^2r)$ out of the denominator so that it can be written

$$\Delta\phi = 2\ell \int_{r_1}^{r_2} \frac{dr}{r^2}\left(1 - \frac{2GM}{c^2r}\right)^{-1/2}\left[c^2e^2\left(1 - \frac{2GM}{c^2r}\right)^{-1} - \left(c^2 + \frac{\ell^2}{r^2}\right)\right]^{-1/2}.$$

The factor in the brackets is then still the square root of a quantity quadratic in $1/r$ to order $1/c^2$. To derive the expression I quoted in class, $\delta\phi_{\text{prec}} = 6\pi\left(\frac{GM}{c\ell}\right)^2$, evaluate the expression above as follows:

(**a**) Expand the factors of $(1-2GM/c^2r)$ in the preceding equation in powers of $1/c^2$, keeping only the $1/c^2$ corrections to Newtonian quantities and using Hartle's Eq. (9.53).

(**b**) Introduce the integration variable $u = 1/r$ and show that the integral can be put in the form

$$\Delta\phi = \left[1 + 2\left(\frac{GM}{c\ell}\right)^2\right] 2\int_{u_2}^{u_1} \frac{du}{[(u_1-u)(u-u_2)]^{1/2}} + \frac{2GM}{c^2}\int_{u_2}^{u_1} \frac{udu}{[(u_1-u)(u-u_2)]^{1/2}} + \cdots,$$

where the dots represent higher order corrections in $1/c^2$.

(**c**) The first integral (including the 2) is just Hartle's Eq. (9.54) and equals 2π. Show that the second integral gives $(\pi/2)(u_1+u_2)$ and that this equals $\pi GM/\ell^2$ to lowest order in $1/c^2$.

(**d**) Combine these results to derive the result we discussed in class $\delta\phi_{\text{prec}} = 6\pi\left(\frac{GM}{c\ell}\right)^2$.