Homework 9

Due Friday, April 15th at 5pm

Read Chapter 20 of Hartle's *Gravity*.

1. Check that the normal vector to the horizon three-surface of a Schwarzschild black hole is a null vector.

2. Consider the spacetime specified by the line element

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

   Except for $r = M$, the coordinate t is always timelike and the coordinate r is spacelike.

   (**a**) Find a transformation to new coordinates (v, r, θ, ϕ) analogous to Hartle's Eq. (12.1) that sets $g_{rr} = 0$ and shows that the geometry is not singular at $r = M$.

   (**b**) Sketch a (t, r) diagram analogous to Hartle's Figure 12.2 showing the world lines of ingoing and outgoing light rays and the light cones.

   (**c**) Is this the geometry of a black hole?

3. An observer falls radially into a spherical black hole of mass M. The observer starts from rest relative to a stationary observer at a Schwarzschild coordinate radius of $10M$. How much time elapses on the observer's own clock before hitting the singularity?

4. An observer decides to explore the geometry outside a Schwarzschild black hole of mass M by starting with an initial velocity at infinity and then falling freely on an orbit that will come close to the black hole and then move out to infinity again. What is the closest that the observer can come to the black hole on an orbit of this kind? How can the observer arrange to have a long time to study the geometry between crossing the radius $r = 3M$ and crossing it again?

5. On Monday we will show in class that when a 4-vector $\mathbf{a}$ is considered in two different coordinate systems x^α and x'^α that its components in the two systems are connected by the transformation law

$$a'^\beta = \frac{\partial x'^\beta}{\partial x^\alpha} a^\alpha.$$

   (**a**) Show explicitly that this transformation rule leads to the transformation of vector components under a Lorentz boost, such as Hartle's Eq. (4.33), given in Hartle's Eq. (5.9). This finally justifies the claim that I've been making since the beginning of the course that every 4-vector transforms under boosts in the same was as the position-time 4-vector.

   (**b**) Use the transformation that connects rectangular coordinates (t, x, y, z) for flat space to polar coordinates (t, r, θ, ϕ) to find the explicit transformation laws giving the components (a^t, a^x, a^y, a^z) of a vector $\mathbf{a}$ in terms of the components $(a^t, a^r, a^\theta, a^\phi)$ and the components $(a^t, a^r, a^\theta, a^\phi)$ in terms of (a^t, a^x, a^y, a^z).