General Relativity

Day 10

Feb 22nd, 2016

__Today__

I Last time

II S.R. Dynamics

I We studied the kinematics of special relativity:

$$x^\alpha = x^\alpha(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))$$

$$u^\alpha \overset{\text{def}}{\equiv} \frac{dx^\alpha}{d\tau} = (\gamma, \gamma \vec{v}).$$

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1 \quad \text{always.}$$

$$a^\alpha \overset{\text{def}}{\equiv} \frac{du^\alpha}{d\tau} \quad \text{and} \quad \underset{\sim}{a} \cdot \underset{\sim}{u} = 0 \quad \text{always!}$$

II __Newton's 1st law__ carries over in the form,

$$\frac{d\underset{\sim}{u}}{d\tau} = 0$$

$\Rightarrow$ straight line motion when there are no forces and $\vec{v} = \text{const.}$

__Newton's 2nd law__:

$$\underset{\sim}{f} = m\frac{d\underset{\sim}{u}}{d\tau} = m\underset{\sim}{a} = \frac{d\underset{\sim}{p}}{d\tau},$$

We call $\underset{\sim}{f}$ the four-force (or the Minkowski force).

$$p^\alpha \overset{\text{def}}{\equiv} m u^\alpha = (E, \vec{p})$$

Relativistic energy & momentum:

$$E = \gamma m$$

$$\vec{p} = \gamma m \vec{v}$$

and

$$\underset{\sim}{p} \cdot \underset{\sim}{p} = -m^2 \iff E^2 = m^2 + p^2$$

Finally, note that

$$\frac{\vec{p}}{E} = \frac{\gamma m \vec{v}}{\gamma m} = \vec{v},$$

which can be very useful in calculations!

Notice that

$$\underset{\sim}{f} \cdot \underset{\sim}{u} = m \underset{\sim}{a} \cdot \underset{\sim}{u} = 0.$$

So, only three of these equations ($\underset{\sim}{f} = m\underset{\sim}{a}$) are independent.

~o~

Despite its hybrid character it is useful to also introduce the three-force

$$\vec{F} = \frac{d\vec{p}}{dt} \begin{matrix} \leftarrow \text{relativistic momentum} \\ \leftarrow \text{coordinate time} \end{matrix}$$

Note that

$$\frac{dp^0}{d\tau} = \frac{dp^0}{dt}\frac{dt}{d\tau} = \gamma \frac{dE}{dt} = f^0 = \gamma \vec{F} \cdot V$$

$$\Rightarrow \quad \frac{dE}{dt} = \vec{F} \cdot \vec{V}$$

So, we have a relativistic notion of power — and it is no surprise that this follows from the other equations of motion (E.O.M.) just as it does in standard mechanics.

In terms of the 3-force we have

$$\vec{f} = \frac{d\vec{p}}{d\tau} = \frac{d\vec{p}}{dt}\frac{dt}{d\tau} = \gamma \vec{F}$$

Also,

$$\underset{\sim}{f} \cdot \underset{\sim}{u} = 0 \Rightarrow -f^t \cdot \gamma + \gamma^2 \vec{F} \cdot \vec{V} = 0$$

$$\Rightarrow \quad f^t = \gamma \vec{F} \cdot \vec{V}$$

So,

$$\boxed{f^\mu = (\gamma \vec{F} \cdot \vec{V}, \gamma \vec{F})}$$

<u>Example</u>: Find $x(t)$ for motion in one dimension under a constant "three"-force F.

Say it starts at $x = x_0$, from rest at $t = 0$. In general

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow \quad dp = F\,dt \overset{\text{w/ b.c.}}{\Rightarrow} p = Ft$$

Then

$$\gamma m V = Ft \Rightarrow \frac{mV}{\sqrt{1 - V^2}} = Ft$$

Solving this for V,

$$\frac{m^2 V^2}{1 - V^2} = F^2 t^2$$

$$\Rightarrow \quad m^2 V^2 = F^2 t^2 - F^2 t^2 V^2$$

$$\Rightarrow \quad V^2 (m^2 + F^2 t^2) = F^2 t^2$$

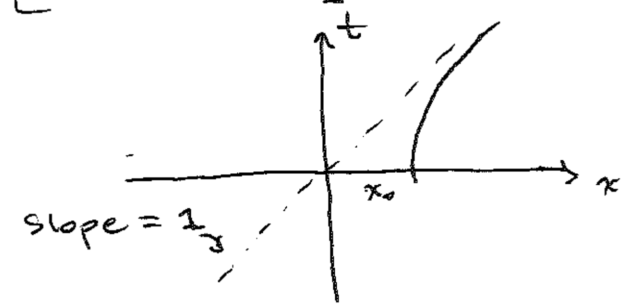
$$\Rightarrow \quad V = \frac{F t}{\sqrt{m^2 + F^2 t^2}}$$

If we separate variables and integrate again,

$$\int_{x_0}^{x} dx' = \int_0^t \frac{F \tilde{t}}{\sqrt{m^2 + F^2 \tilde{t}^2}} \, d\tilde{t}$$

$$\Rightarrow \quad x(t) - x_0 = \frac{1}{F} \sqrt{m^2 + F^2 t^2} - \frac{m}{F}$$

$$\text{or} \quad \left[x(t) - x_0 + \frac{m}{F} \right]^2 = \frac{m^2}{F^2} + t^2$$



We can also see what's happening with the speed: