General Relativity

Day 11

Feb 24th, 2016

<u>Today</u>

I Last time

II Measurement

III Light's Energy & momentum

I. Newton:

1st $\frac{d\underset{\sim}{u}}{d\tau} = 0 \iff$ Straight line motion & $\vec{v} = \text{const.}$

2nd $\underset{\sim}{f} = m\frac{d\underset{\sim}{u}}{d\tau} = m\underset{\sim}{a} = \frac{d\underset{\sim}{p}}{d\tau}$

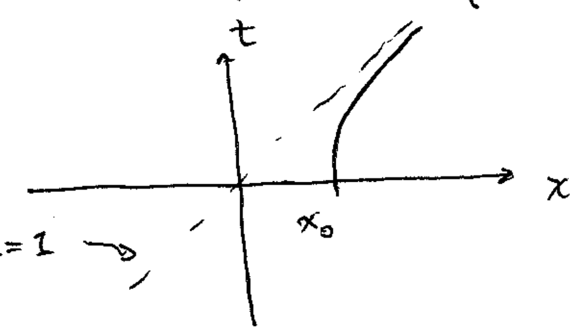
4-force or Minkowski force

3-force: $\vec{F} = \frac{d\vec{p}}{dt}$

$$f^{\mu} = (\gamma \vec{F}\cdot\vec{v}, \gamma \vec{F}).$$

Ex: Constant 3-force in 1D:

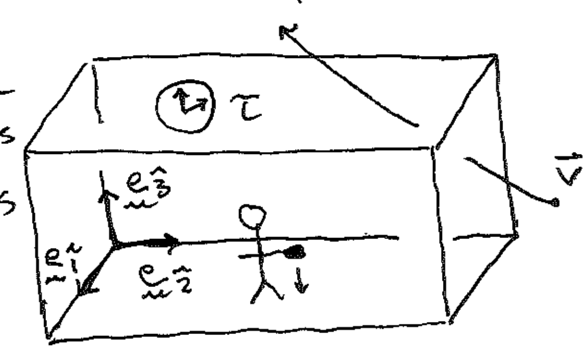
$$\left[x(t) - x_0 + \frac{m}{F}\right]^2 = \frac{m^2}{F^2} + t^2$$



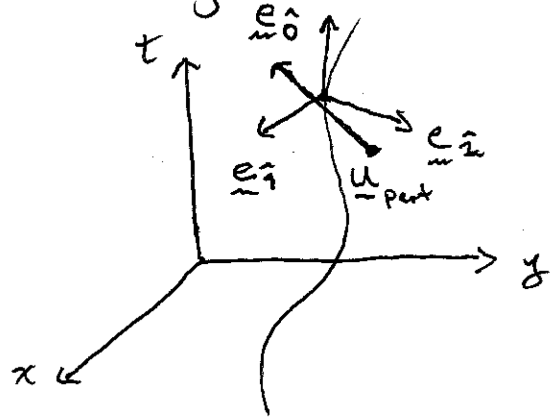
$$V(t) = \frac{Ft}{\sqrt{m^2 + F^2 t^2}}$$

II The relational aspect of both S.R. and G.R. can make measurements difficult to interpret. However, there is a very general strategy that is also quite natural operationally.

Observers make local measurements by setting up axes and clocks: four orthogonal unit vectors $\underset{\sim}{e}_{\hat{0}}, \underset{\sim}{e}_{\hat{1}}, \underset{\sim}{e}_{\hat{2}}, \underset{\sim}{e}_{\hat{3}}$.



Spacetime diagram



Recall $\underset{\sim}{u}_{obs} \cdot \underset{\sim}{u}_{obs} = -1$, so $\underset{\sim}{u}_{obs}$ is a unit (timelike) vector. So an observer naturally chooses

$$\underset{\sim}{e}_{\hat{0}} = \underset{\sim}{u}_{obs}$$

This captures the observation that we are always at rest in our own frame, but moving in time.

To determine what an observer measures we decompose, e.g., a particle's 4-momentum $\underset{\sim}{p}$ into the observer's frame:

$$\underset{\sim}{p} = p^{\hat{\alpha}} \underset{\sim}{e}_{\hat{\alpha}}$$

The $p^{\hat{\alpha}}$ represent the energy ($p^{\hat{0}}$) and momentum ($p^{\hat{i}}$) as measured by our moving observer. To calculate these components we use

$$\underset{\sim}{e}_{\hat{\beta}} \cdot \underset{\sim}{p} = p^{\hat{\alpha}} \underbrace{\underset{\sim}{e}_{\hat{\beta}} \cdot \underset{\sim}{e}_{\hat{\alpha}}}_{\text{assume these are orthonormal}}$$

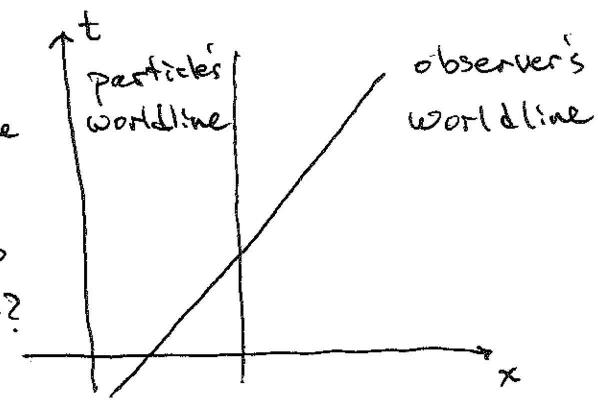
$$= p^{\hat{\alpha}} \eta_{\hat{\beta}\hat{\alpha}}$$

$$\Rightarrow p^{\hat{0}} = -\underset{\sim}{e}_{\hat{0}} \cdot \underset{\sim}{p}, \quad p^{\hat{1}} = \underset{\sim}{e}_{\hat{1}} \cdot \underset{\sim}{p}, \text{ etc.}$$

Let's see it work in practice:

<u>Ex.</u> Consider a particle, mass m, at rest in an inertial frame. An observer moves along the x-axis of this frame with velocity $\vec{v}$.

What is the energy of the particle according to this observer?



Three different methods:

(1) According to the observer the particle moves with velocity $-\vec{V}$ and so,

$$E = \gamma m \,, \qquad \gamma = \gamma(|\vec{V}|).$$

(2) In the frame drawn

$$\underset{\sim}{u}_{obs} = (\gamma, \gamma V, 0, 0) = \underset{\sim}{e}_{\hat{0}} \,, \quad \gamma = \gamma(|\vec{v}|)$$

$$\underset{\sim}{p} = (m, 0, 0, 0)$$

and so,

$$E = -\underset{\sim}{e}_{\hat{0}} \cdot \underset{\sim}{p} = -(-m\gamma) = m\gamma. \checkmark$$

(3) In the observer's frame

$$\underset{\sim}{u}_{obs} = (1, 0, 0, 0) = \underset{\sim}{e}_{\hat{0}}$$

$$\underset{\sim}{p} = (\gamma m, -\gamma m V, 0, 0)$$

and so,

$$E = -\underset{\sim}{e}_{\hat{0}} \cdot \underset{\sim}{p} = \gamma m. \checkmark$$

Methods (2) and (3) illustrate an important point: Dot products are Lorentz invariants and so you can evaluate them in any convenient coordinates — you will always get the same answer. Method (1) will become difficult in G.R. and methods (2) & (3) become far superior.

III Light has the amazing property $m = 0$ and travels at $V = 1$. Einstein recognized that for a photon

$$E = \hbar\omega$$

angular frequency

Planck's const./2π = 10^{-34} J.s

The energy-momentum relation

$$E^2 = m^2 + \vec{p}^{\,2}$$

Continues to hold for light w/ $m = 0$,

$$\Rightarrow \boxed{|\vec{p}| = E}$$

Because $E = \hbar\omega$ it is convenient to factor out an $\hbar$ from $\vec{p}$ too!

$$\vec{p} = \hbar\vec{k}$$

and this implies

$$|\vec{p}| = \hbar|\vec{k}| = E = \hbar\omega$$

or

$$\Rightarrow |\vec{k}| = \omega$$