Today

I Last time

II Clean up

III Gravitational Time Dilation as Geometry

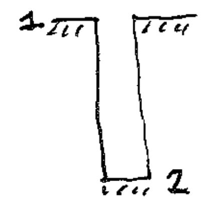
IV Particle Motion in Curved Spacetime

General Relativity Day 14

Mar 3rd, 2016

I We treated gravitational time dilation twice.

- Naively you can drop a photon down a mine shaft!

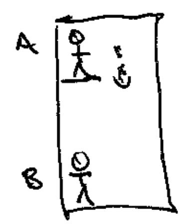


$$\omega_2 = \omega_1 \left(1 + \frac{gh}{c^2}\right)$$

Useful for reconstructing the result quickly.

- A more careful argument using the equivalence principle and a rocket gave;

$$\Delta\tau_B = \frac{\Delta\tau_A}{\left(1 + \frac{gh}{c^2}\right)}$$



- The slogan I use is "Slouching clocks run slow"

II The acceleration g refers to the acceleration due to gravity at the surface of the Earth. We'd like to have a more general expression. We'll use the gravitational potential Φ. (N.B. this is defined as the potential energy per unit mass, i.e. $\Phi = -\frac{GM}{r}$ not $U = -\frac{GMm}{r}$ and $[\Phi] = \frac{\text{Energy}}{\text{mass}}$) units of

So, $gh = \Phi_A - \Phi_B$.

If we also use the reception rates, $\text{rate}_A = 1/\Delta\tau_A$, $\text{rate}_B = 1/\Delta\tau_B$.

then,

$$\text{rate}_B = \text{rate}_A\left(1 + \frac{gh}{c^2}\right)$$

$$= \text{rate}_A\left(1 + \frac{\Phi_A - \Phi_B}{c^2}\right).$$

III We begin our program in earnest:
I hand you a metric, say a little about it and then you explore it.
Our first curved spacetime metric:

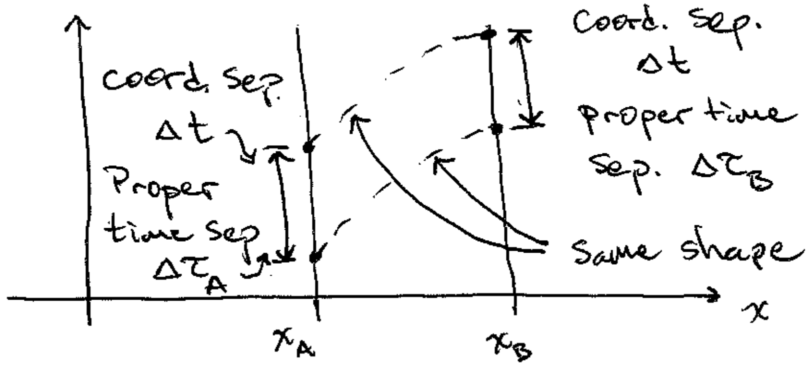
$$ds^2 = -\left(1 + \frac{2\Phi(x^i)}{c^2}\right)(c\,dt)^2 + \left(1 - \frac{2\Phi(x^i)}{c^2}\right)(dx^2 + dy^2 + dz^2)$$

here $\Phi = \Phi(x^i) = \Phi(x,y,z)$ is the gravitational potential in the Newtonian sense just discussed and is a function of <u>position</u> alone.

($\Phi(x^i)$ is shorthand for $\Phi(x,y,z)$.)

We call this a static potential. This line element is only valid for small curvatures and weak sources, which is known as a <u>weak field</u>. So this is the "static weak field metric."

What does light do in this curved geometry? Light rays are <u>not straight lines</u>. But, the metric <u>is</u> static, so the shape of two light rays at different times is the same.



Important calculation: What are $\Delta\tau_A$ and $\Delta\tau_B$ to first order in $\frac{\Phi(x^i)}{c^2}$?

Well,

$$\Delta\tau_A^2 = -\frac{\Delta s^2}{c^2} = \left(1 + \frac{2\Phi_A}{c^2}\right)\frac{c^2\Delta t^2}{c^2},$$

where $\Phi_A \equiv \Phi(x_A)$. Then

$$\Delta\tau_A = \sqrt{1 + \frac{2\Phi_A}{c^2}}\,\Delta t \approx \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t.$$

Very similarly, $\Delta\tau_B \approx \left(1 + \frac{\Phi_B}{c^2}\right)\Delta t$

$$\Rightarrow \Delta\tau_B = \frac{\left(1 + \frac{\Phi_B}{c^2}\right)}{\left(1 + \frac{\Phi_A}{c^2}\right)}\Delta\tau_A$$

But then,

$$\Delta\tau_B \approx \left(1+\frac{\Phi_B}{c^2}\right)\left(1-\frac{\Phi_A}{c^2}\right)\Delta\tau_A$$

$$= \left(1+\frac{\Phi_B-\Phi_A}{c^2}\right)\Delta\tau_A + \cdots$$

Same result as before! What part of the line element mattered most for this calculation? It was the curvature in time that mattered — the spatial metric played no role!

IV How do particles move in a curved spacetime?

<u>Reminder</u>: In mechanics we have

$$S = \int_{t_A}^{t_B} L\,dt\,, \qquad L = T - V$$

and the eq. of motion follow from

$$\delta S = 0 \iff \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

This was an alternative axiomatization to Newton's $\vec{F} = m\vec{a}$.

Our geometrical context makes room for a new answer:

> **Variational Principle for <u>Free</u> Particle Motion**
>
> The world line of a free particle btwn two timelike separated pts extremizes the proper time btwn them.

Let's try it; <u>Example</u>: Flat Minkowski spacetime

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \sqrt{-ds^2}$$

$$= \int_A^B \left(dt^2 - (dx^2+dy^2+dz^2)\right)^{1/2}$$