General Relativity

Mar 4th, 2016

Day 15

Today ~~Last Time~~

II Particle Motion in Curved Spacetime

III Motion in a static weak field metric

I • We cast time dilation in a general form

$$\text{rate}_B = \text{rate}_A \left(1 + \frac{\Phi_A - \Phi_B}{c^2}\right)$$

• Studied the "static, weak field" metric:

$$ds^2 = -\left(1 + 2\Phi\right) dt^2 + \left(1 - 2\Phi\right)\left(dx^2 + dy^2 + dz^2\right)$$

• We noticed that curvature in time dominated the time dilation calculation.

This was an alternate axiomatization to Newton's $\vec{F} = m\vec{a}$.

Our geometrical context makes room for a new answer to the motion question:

II How do particles move in a curved spacetime?

Reminder: In mechanics we have

$$S = \int_{t_A}^{t_B} L\, dt, \qquad L = T - V$$

and the eqs. of motion follow from

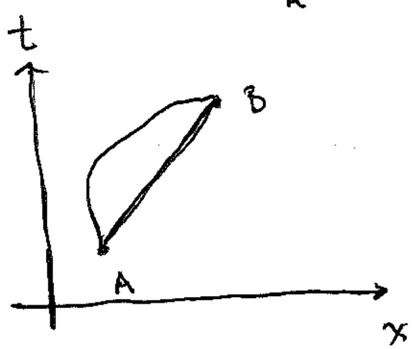
$$\delta S = 0 \iff \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0.$$

$$\boxed{\begin{array}{c}\text{Variational Principle} \\ \text{for } \underline{\text{Free}} \text{ Particle Motion} \\ \text{The worldline of a free particle btwn two timelike separated pt.s extremizes the proper time btwn them.}\end{array}}$$

Let's try it; <u>Example</u>: Flat Minkowski spacetime

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \sqrt{-ds^2}$$

$$= \int_A^B \left(dt^2 - [dx^2+dy^2+dz^2]\right)^{1/2}$$



To do this integral along a worldline, we need to choose a parameter along that worldline, call it σ.

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{dx^\alpha}{d\sigma}\right)}\right) = \frac{\partial L}{\partial x^\alpha} \qquad \left(\text{Recall: } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}\right)$$

Now just calculate; For $x^1 = x$ we get

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{dx}{d\sigma}\right)}\right) = \frac{d}{d\sigma}\left(\frac{1}{2}\frac{1}{L}\cdot\left\{-2\frac{dx}{d\sigma}\right\}\right) = \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{d\sigma}\left(\frac{1}{L}\frac{dx}{d\sigma}\right) = 0$$

But, note that $\frac{1}{L} = \left(\frac{d\tau}{d\sigma}\right)^{-1} = \frac{d\sigma}{d\tau}$, so

$$\Rightarrow \frac{d}{d\sigma}\left(\frac{d\sigma}{d\tau}\frac{dx}{d\sigma}\right) = \frac{d}{d\sigma}\left(\frac{dx}{d\tau}\right) = 0$$

Then if $\sigma = 0$ corresponds to A and $\sigma = 1$ to B. we have

$$\tau_{AB} = \int_0^1 d\sigma \left[\left(\frac{dt}{d\sigma}\right)^2 - \left\{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right\}\right]^{1/2}$$

This means

$$L = \left[\left(\frac{dt}{d\sigma}\right)^2 - \left\{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2\right\}\right]^{1/2} = \frac{d\tau}{d\sigma}$$

and we can use the Euler-Lagrange eqn's to find the worldline that extremizes τ_{AB}:

or

$$\frac{d\sigma}{d\tau}\frac{d}{d\tau}\left(\frac{dx}{d\tau}\right) = 0$$

$$\Rightarrow \boxed{\frac{d^2x}{d\tau^2} = 0}$$

The equation of a straight line in the x coordinate. Same calculation for the other coords gives

$$\frac{d^2x^\alpha}{d\tau^2} = 0$$

and this is a straight worldline.

III What about the static weak field case?

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \left(-\frac{ds^2}{c^2}\right)^{1/2}$$

$$= \int_A^B \left[\left(1+\frac{2\Phi}{c^2}\right)dt^2 - \frac{1}{c^2}\left(1-\frac{2\Phi}{c^2}\right)(dx^2+dy^2+dz^2)\right]^{1/2}$$

Let's choose t as our parameter:

$$\tau_{AB} = \int_A^B dt\left\{\left(1+\frac{2\Phi}{c^2}\right) - \frac{1}{c^2}\left(1-\frac{2\Phi}{c^2}\right)\underbrace{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]}_{=|\vec{v}|^2}\right\}^{1/2}$$

$$\approx \int_A^B dt\left\{\left(1+\frac{2\Phi}{c^2}\right) - \frac{|\vec{v}|^2}{c^2}\right\}^{1/2} \quad \left(\text{dropping } \frac{2\Phi|\vec{v}|^2}{c^4}\right)$$

$$\approx \int_A^B dt\left\{1 - \frac{1}{c^2}\left(\frac{1}{2}|\vec{v}|^2 - \Phi\right)\right\} \quad (\text{Taylor expanding})$$

This is precisely Newton's universal law of gravitation (check it)!

Interesting remark: Notice that for both the gravitational time dilation and the recovery of the Newtonian equations of motion it was the curvature in time that was most important; the spatial coefficient $\left(1-\frac{2\Phi}{c^2}\right)$ was irrelevant.

So, neglecting the constant $+1$,

$$L = \frac{1}{2}\vec{v}^2 - \Phi(x^i), \quad i=1,2,3$$

the usual Lagrangian divided by m. What are the E-L eqn.s?

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\left(\frac{dx}{dt}\right)}\right) = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{\partial L}{\partial x} = -\frac{\partial \Phi}{\partial x}$$

or for all the components

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\Phi$$