Today

General Relativity Day 16

Mar 7th, 2016

I Coordinates

II Coordinates: Bugaboos
(= "A fancied object of terror")

III Fitting it all together: Local Inertial Frames (LIFs)

I Even more than in your past experiences, no one coordinate system is good for all of G.R. Instead we try to extract physically meaningful quantities that are independent of the coords used to describe them.

<u>Coords</u>: A unique set of labels for every point in the region they cover.

<u>Ex.</u>: Region: Minkowski (flat) spacetime

Coord.s: (t, x, y, z), $-\infty < t, x, y, z < \infty$ (Range)

or

Coord.s: (t, r, θ, ϕ) $-\infty < t < \infty$

$0 < r < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$

and many, many more.

One reason for our focus on invariants is that the <u>same</u> metric can look very different in different coordinates

Minkowski metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

In spherical coord.s:

$x = r s\theta c\phi$, $y = r s\theta s\phi$, $z = r c\theta$

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

If you hadn't seen this transform a number of times it could be difficult to recognize these as the same metric. More on this shortly.

<u>Metric in general coord.s</u>: For general coord.s x^α, $\alpha = 0, 1, 2, 3$, we have

$$\boxed{ds^2 = g_{\alpha\beta}(x)\, dx^\alpha dx^\beta}$$

and $g_{\alpha\beta}$ is called the metric.

<u>Ex.</u>: Flat spacetime in spherical coord.s

$$g_{\alpha\beta}(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 s^2\theta \end{pmatrix}$$

$$= \text{Diag}(-1, 1, r^2, r^2 s^2\theta).$$

The metric is symmetric and so it has 10 independent components. Due to coord. transformations 4 of these are arbitrary and so there are 6 true freedoms.

II. Most coordinate systems have issues. For example, in spherical coords, $\theta = 0$ and $\theta = \pi$ for fixed (t, r) don't have unique labels (all values of ϕ correspond to the same point). Mathematicians have fixed this sort of a problem with the notion of a manifold. Another example in the plane is

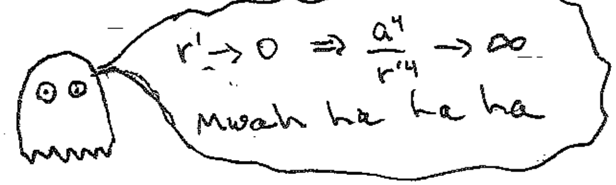
$$dS^2 = dr^2 + r^2 d\phi^2$$

under the transformation $r = \frac{a^2}{r'}$, with $a = \text{const}$.

$$dr = -\frac{a^2}{r'^2} dr'$$

$$\Rightarrow ds^2 = \frac{a^4}{r'^4} dr'^2 + \frac{a^4}{r'^2} d\phi^2$$

$$= \frac{a^4}{r'^4}\left(dr'^2 + r'^2 d\phi^2\right)$$



At $r'=0$ the line element blows up. This is our fault for choosing a "bad" coordinate transformation. But it can be (and was) quite confusing at first. As we will see black hole geometries have these coordinate singularities. This caused considerable confusion in the early days of G.R.

The result: Start with a general metric $g_{\alpha\beta}$ in arbitrary coordinates

$$g_{\alpha\beta} = g_{\alpha\beta}(x)$$

Choose a point P of interest then We can always choose new coordinates x' such that

$$g'_{\alpha\beta}(x'_P) = \eta_{\alpha\beta}$$

Q: Can we do even better? A: Slightly.

III Local Inertial Frames

The transition to general coordinates is unsettling. How do we use anything from our old tool box (S.R. etc)? The answer to this question is an elegant synthesis of everything we've been thinking about:

We can choose our coord.s s.t.

$$\boxed{g'_{\alpha\beta}(x'_P) = \eta_{\alpha\beta} \quad \text{and} \quad \left.\frac{\partial g'_{\alpha\beta}}{\partial x'^{\gamma}}\right|_{x=x_P} = 0}$$

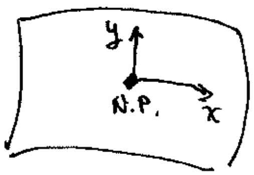
We call these coordinates a locally inertial frame at P (or Riemann normal coordinates). This is precisely what our observers were doing when we discussed (local) measurement. This is how special relativity fits into general relativity. This is one more precise formulation of the content of the equivalence principle.

<u>How does it work?</u> As we've noted $g_{\alpha\beta}$ is a symmetric matrix. We choose our coordinate transformation so that this matrix is diagonalized at P

<u>Wonderful Example</u>: We all know that local observers once thought the Earth was flat. What coord.s were they using?

$$ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2) \quad \text{(sphere)}$$

slice of sphere near north pole (N.P.)



then by rescaling the new coordinates we can bring the diagonal entries to diag(-1, 1, 1, 1).

We'll discuss the additional adjustments necessary for $\partial g'_{\alpha\beta} / \partial x'^{\gamma} \big|_{x_P} = 0$ in a few weeks.

Note that while x' coords achieve this at P they do not in general achieve it at other points.

Local observers use

$$x = \underbrace{a\theta}_{\text{arc length}} \underbrace{\cos\phi}_{x\text{-component}}, \quad y = a\theta\sin\phi$$

1st derivative vanishes at $(x,y) = (0,0)$

$$g_{AB}(x,y) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{flat metric}} + \begin{pmatrix} -\frac{2y^2}{3a^2} & \frac{2xy}{3a^2} \\ \frac{2xy}{3a^2} & -\frac{2x^2}{3a^2} \end{pmatrix}$$

+ higher order

$A, B = (1, 2)$.

Check this!