Today

I Last time

II Vectors in Curved Geometries

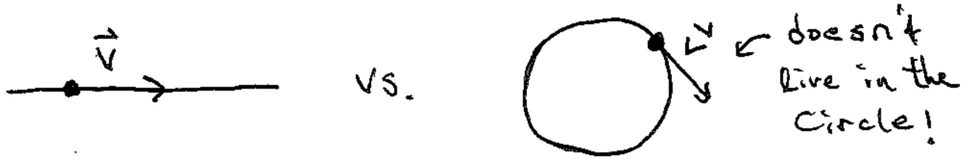
I
- Intro'd metric in general coord.s x^α,

$$\boxed{ds^2 = g_{\alpha\beta}(x)\,dx^\alpha dx^\beta}$$

- Coordinate singularities
- Local Inertial Frames (LIFs) (Riemann Normal Coordinates)

$$g_{\alpha\beta}(x_p) = \eta_{\alpha\beta}, \quad \left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_{x=x_p} = 0$$

II There is a subtlety you are already familiar with but may not have noticed: for curved geometries vectors don't live in position space! Consider a bead on a wire



We have to address several questions then:

(1) How does a local observer talk about (i.e.) measure vectors?

A: The key is to separate directions and magnitudes. Directions are accesible locally and then we <u>impose</u> linearity, i.e.

$$\alpha(\underline{a} + \underline{b}) = \alpha\underline{a} + \alpha\underline{b}$$

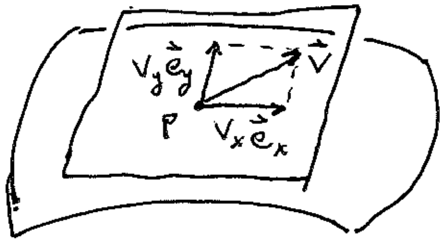
to build up larger magnitude vectors.

(Velocity is a great example to keep in mind. Your heading is the direction and the magnitude of the derivative in that direction is the speed.)

This leads to (2) Where do vectors live?

The local construction just described is what mathematicians call a tangent space.

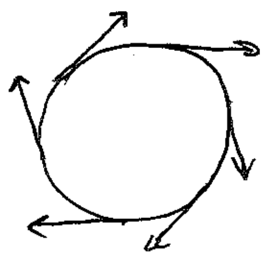
Pictorially



Vectors live in the tangent space to a point.

(3) Does this mean that vectors based at different points of our space live in different tangent spaces? A: Yes! This is important because we can't perform vector operations on vectors that live in different tangent spaces.

A vector field is an assignment of a vector to each point of your space, e.g.



We denote this by $\underset{\sim}{a}(x)$.

Now, recall our calculation

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = (a^\alpha \underset{\sim}{e}_\alpha)\cdot(b^\beta \underset{\sim}{e}_\beta) = a^\alpha b^\beta (\underset{\sim}{e}_\alpha\cdot\underset{\sim}{e}_\beta)$$

Depending on what basis we choose this calculation has different characters.

Bases: With these new insights we need to return to the notion of a basis.

<u>Orthonormal Basis:</u>

A basis for which

$$\boxed{\underset{\sim}{e}_{\hat{\alpha}}(x)\cdot\underset{\sim}{e}_{\hat{\beta}}(x) = \eta_{\hat{\alpha}\hat{\beta}}}$$

hatted indices tell us that we're working with an orthonormal basis.

In such a basis

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = \eta_{\hat{\alpha}\hat{\beta}}\, a^{\hat{\alpha}} b^{\hat{\beta}}$$

As you know, these bases are important because they are the bases of potential observers

$$\underset{\sim}{p} = p^{\hat{\alpha}}\, \underset{\sim}{e}_{\hat{\alpha}}$$

these components are observable.

Coordinate Basis:

We've worked with

$$u^\alpha = \frac{dx^\alpha}{d\tau}$$

extensively. Implicitly this was in a particular basis. Which one? Well, certainly $\underset{\sim}{u} \cdot \underset{\sim}{u} = -1$, and for

$$g_{\alpha\beta}\, u^\alpha u^\beta = g_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

But,

$$g_{\alpha\beta}\, dx^\alpha dx^\beta = ds^2 = -d\tau^2$$

so,

$$g_{\alpha\beta}\, u^\alpha u^\beta = -1$$

Then these components must be in a basis such that

coord. basis $\boxed{\underset{\sim}{e}_\alpha(x) \cdot \underset{\sim}{e}_\beta(x) = g_{\alpha\beta}(x)}$