General Relativity

Mar 11th, 2016

Day 18

<u>Today</u>

I Last time

II The Geodesic Equation

I
- Vectors in a curved geometry
  (1) Measure direction locally & impose linearity
  (2) They live in the tangent space
  (3) Can only apply vector operations in one tangent space.
- Usually we calculate with coordinate bases ($\underset{\sim}{e}_\alpha$) and interpret with orthonormal bases ($\underset{\sim}{e}_{\hat{\alpha}}$).

II In practice how will we explore or discover the geometry of a spacetime?

Idea: Release a bunch of free test particles and allow them to probe the geometry directly by traveling on geodesics. So, we turn to the study of the geodesic eqn., our first fundamental eqn. in G.R.

Variational Principle for Free Test Particle Motion

The world line of a <u>free</u> <u>test</u> particle between two timelike separated points extremizes the proper time between them.

<u>test particle</u>: This is a particle with a small enough mass that its effect on the curvature of spacetime can be neglected.

<u>free particle</u>: free of any influences except gravity.

Trajectories that extremize proper time are called <u>geodesics</u>. In space these are curves that extremize the length of the curve.

Then the E-L equations are,

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial\left(\frac{dr}{d\sigma}\right)}\right) = \frac{d}{d\sigma}\left(\frac{1}{L}\frac{dr}{d\sigma}\right) \qquad r\text{-eq.}$$

$$= \frac{\partial L}{\partial r} = \frac{r}{L}\left(\frac{d\phi}{d\sigma}\right)^2$$

$$\frac{d}{d\sigma}\left(\frac{1}{L}r^2\frac{d\phi}{d\sigma}\right) = 0 \qquad \phi\text{-eq.}$$

Note that, $L = \frac{dS}{d\sigma}$

<u>Example</u>: Plane in polar coord.s

$$dS^2 = dr^2 + r^2 d\phi^2$$

parametrize curve by σ, i.e. give $r(\sigma)$ and $\phi(\sigma)$ then

$$S_{AB} = \int_A^B dS = \int_A^B (dr^2 + r^2 d\phi^2)^{1/2}$$

$$= \int_0^1 d\sigma \underbrace{\left[\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\phi}{d\sigma}\right)^2\right]^{1/2}}_{L\left(r, \frac{dr}{d\sigma}, \frac{d\phi}{d\sigma}\right)}$$

This means that if we parametrize by S we can get rid of the messy Ls:

$$\boxed{\begin{aligned} r\text{-eq}: &\quad \frac{d}{dS}\left(\frac{dr}{dS}\right) = \frac{d^2r}{dS^2} = r\left(\frac{d\phi}{dS}\right)^2 \\ \phi\text{-eq}: &\quad \frac{d}{dS}\left(r^2\frac{d\phi}{dS}\right) = 0. \end{aligned}}$$

<u>In fact</u>, this works in general — you can <u>drop</u> <u>the squareroot</u> inside

the action whenever you parametrize by proper time (or arc length for spacelike geodesics). See proof in supplementary notes if you like.

<u>Note</u>: This doesn't work in the same manner for lightlike geodesics because

$$d\tau^2 = -ds^2 = 0\,!$$

variational principle. We would like to extremize

$$\tau = \int d\tau = \int \sqrt{-ds^2}$$

(Recall $d\tau^2 = -ds^2$). But, in a general geometry

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta,$$

so

$$\tau = \int \sqrt{-g_{\alpha\beta}\, dx^\alpha dx^\beta}.$$

Moral: drop the square root whenever you can manage to make this work. (We'll illustrate this again momentarily.)

<u>Geodesics in general</u>:

To derive the geodesic eqn.s in general we turn to the

Parametrizing this integral by proper time itself, we get

$$\tau = \int \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}\; d\tau.$$

With this choice we can use the "drop the square root" rule and the Lagrangian for which we would like to derive the EL eqn.s is

$$\boxed{L = -g_{\alpha\beta}(x) \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}.}$$