General Relativity

Day 19 — Mar 14th, 2016

Today

I Last time

II Deriving the geodesic equation

III Example: Plane in polar coords

II To proceed we calculate the EL eqns for this Lagrangian:

$$\frac{\partial L}{\partial x^\delta} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\delta}\right) = 0.$$

Taking these derivatives

$$-\frac{\partial g_{\beta\gamma}}{\partial x^\delta}\dot{x}^\beta\dot{x}^\gamma - \frac{d}{d\tau}\left(-g_{\beta\gamma}\,\delta^\beta_\delta\,\dot{x}^\gamma - g_{\beta\gamma}\,\dot{x}^\beta\,\delta^\gamma_\delta\right) = 0$$

where we have expressed the derivative

$$\frac{\partial \dot{x}^\alpha}{\partial \dot{x}^\delta} = \delta^\alpha_\delta$$

I
- Learned that we can drop the square root in the calculus of variations, provided we choose the proper parameter.
- Showed that the Lagrangian for a general timelike geodesic is

$$L = -g_{\beta\gamma}(x)\,\dot{x}^\beta\dot{x}^\gamma, \quad \dot{x}^\alpha \equiv \frac{dx^\alpha}{d\tau}.$$

in terms of the Kronecker δ:

$$\delta^\alpha_\delta = \begin{cases} 1 & \text{if } \alpha = \delta \\ 0 & \text{if } \alpha \neq \delta. \end{cases}$$

This is reasonable since,

$$\frac{\partial \dot{x}^1}{\partial \dot{x}^1} = 1 \quad \text{and} \quad \frac{\partial \dot{x}^0}{\partial \dot{x}^1} = \frac{\partial \dot{x}^2}{\partial \dot{x}^1} = \frac{\partial \dot{x}^3}{\partial \dot{x}^1} = 0.$$

Kronecker δ's are particularly nice in sums, where they simply replace one index by another

eg. $\sum_\alpha \dot{x}^\alpha \delta^\beta_\alpha = \dot{x}^\alpha \delta^\beta_\alpha = \dot{x}^\beta$ ← Prove it!

Returning to our EL equations:

$$\frac{d}{d\tau}\left(g_{\delta\gamma}\dot{x}^{\gamma} + g_{\beta\delta}\dot{x}^{\beta}\right) - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\dot{x}^{\beta}\dot{x}^{\gamma} = 0$$

Next we evaluate the τ-derivative, using the chain rule on $g_{\alpha\beta}(x)$,

$$\frac{\partial g_{\delta\gamma}}{\partial x^{\beta}}\dot{x}^{\beta}\dot{x}^{\gamma} + \frac{\partial g_{\beta\delta}}{\partial x^{\gamma}}\dot{x}^{\beta}\dot{x}^{\gamma} + g_{\delta\gamma}\ddot{x}^{\gamma}$$
$$+ g_{\beta\delta}\ddot{x}^{\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\dot{x}^{\beta}\dot{x}^{\gamma} = 0$$

divide by 2 we get

$$g_{\delta\beta}\ddot{x}^{\beta} + \frac{1}{2}\left(\frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\right)\dot{x}^{\beta}\dot{x}^{\gamma} = 0$$

Finally, introduce $g^{\alpha\delta}$ the inverse of $g_{\delta\beta}$, i.e. $g^{\alpha\delta}g_{\delta\beta} = \delta^{\alpha}_{\beta}$ (think of the matrix inverse.).

Then

$$\ddot{x}^{\alpha} + \underbrace{\frac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}}\right)}_{\text{call this } \Gamma^{\alpha}_{\beta\gamma} \text{ "Christoffel symbols"}}\dot{x}^{\beta}\dot{x}^{\gamma} = 0$$

where we have exploited our freedom to name a dummy index anything we want. The last term of the 1st line and first term of the 2nd line are equal due to the symmetry of the metric and the dummy index renaming freedom. If we collect all the $\dot{x}\dot{x}$ terms and

or

$$\boxed{\ddot{x}^{\alpha} + \Gamma^{\alpha}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma} = 0.}$$

Recall, $\dot{x}^{\alpha} = \frac{dx^{\alpha}}{d\tau} = u^{\alpha}$, so it is also nice to write

$$\boxed{\frac{du^{\alpha}}{d\tau} + \Gamma^{\alpha}_{\beta\gamma}u^{\beta}u^{\gamma} = 0}$$ Geodesic Eq for timelike geodesics

III Let us return to our example of the plane in polar coords: we can read off the components of the Christoffel symbols from the eqn.s of motion. First let's recall some analogies

Timelike		Spacelike
τ	$\longrightarrow$	S arc length
$u^\alpha = \dfrac{dx^\alpha}{d\tau}$	$\longrightarrow$	$u^A = \dfrac{dx^A}{dS}$, $A = 1, 2$
$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1$	$\longrightarrow$	$\vec{u} \cdot \vec{u} = 1$

Because of spacelike $\vec{u}$.

By looking at the appropriate $x^A = (x^1, x^2) = (r, \phi)$ we can read off the Christoffel symbols through this comparison:

$$\Gamma^r_{\phi\phi} = -r$$

while $\Gamma^r_{rr} = \Gamma^r_{r\phi} = \Gamma^r_{\phi r} = 0$. While

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \quad \text{and} \quad \Gamma^\phi_{\phi\phi} = \Gamma^\phi_{rr} = 0.$$

Compare how efficient this is to the direct computation using

The general geodesic eqn is, after these analogies,

$$\frac{d^2x^A}{dS^2} + \Gamma^A_{BC} \frac{dx^B}{dS} \frac{dx^C}{dS} = 0,$$

Compare this to our previous results

r-eq. $\dfrac{d^2r}{dS^2} - r \dfrac{d\phi}{dS} \dfrac{d\phi}{dS} = 0$

ϕ-eq. $\dfrac{d^2\phi}{dS^2} + \dfrac{1}{r} \dfrac{dr}{dS} \dfrac{d\phi}{dS} + \dfrac{1}{r} \dfrac{d\phi}{dS} \dfrac{dr}{dS} = 0$

$$\Gamma^A_{BC} = \frac{1}{2} g^{AD} \left(\frac{\partial g_{DC}}{\partial x^B} + \frac{\partial g_{BD}}{\partial x^C} - \frac{\partial g_{BC}}{\partial x^D} \right)$$

and

$$g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad (A, B = 1, 2).$$

First the inverse metric is

$$g^{AD} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

then

$$\Gamma^r_{\phi\phi} = \frac{1}{2} g^{rr} \left(\overset{0}{\cancel{\frac{\partial g_{r\phi}}{\partial x^\phi}}} + \overset{0}{\cancel{\frac{\partial g_{\phi r}}{\partial x^\phi}}} - \frac{\partial g_{\phi\phi}}{\partial x^r} \right)$$

$$= 1/2 \cdot 1 \cdot (-2r) = -r$$

Correct, but lots of work!