General Relativity
Day 2

Feb 3rd, 2016

Today

I Last time

II Class vote

III The left hand side

IV What is geometry?

I
- We argued that we cannot treat gravity in the same way as E & M because mass-energy is not additive.
- Collecting all forms of mass-energy into a tensor $T^{\mu\nu}$ we guessed that

Einstein's equation should have the form

$$\boxed{???} = \overset{\text{const. containing } G}{k}\, T^{\mu\nu}$$

- Reminder our 4th hour will be Th 8-9pm with my office hours Th 9-10pm and F 10-11am.
- A remarkabe 'coincidence'

$$\cancel{m}\vec{a} = -G\frac{\cancel{m}M}{r^2}\hat{r}$$

$\boxed{\text{Inertial mass = gravitational mass}}$

⇒ Nothing about the body determines how it falls under gravity. So, what does?

II Vote on course tracks:

Track 1 (T1) Black Holes

Track 2 (T2) Gravitational Waves

Track 3 (T3) Cosmology

Results: T1: T2: T3:

Grade: Att. 5%, HW 35%, Quiz 5%, In class 25%, Take Home 30%

Travel

Mathematica

Provisions & lateness

Sign Syllabus

III Einstein's answer to the last question of Sec. I: gravity is not a force at all, but a feature of space and time.

The theory of curvature was developed by Gauss and then Riemann and is part of Differential Geometry.

IV What is geometry?

Egyptian Surveyors measured the land around the Nile. Became the study of shape, size, position, and the properties of Space.

Matter (stress-energy) curves Space & time ⇝ small (test) particles move on the shortest, possible lines (geodesics) in this curved spacetime. (Generalization of Newton's 1st law)

Then

$$\boxed{???} = \kappa T^{\mu\nu}$$

some measure of curvature

Let's discuss two revolutions:

$$\boxed{\text{There exists more than one geometry!}}$$

(Lobachevsky, Bolyai, Gauss)

The Egyptians treated Earth as flat, making errors of order

$$\epsilon = \frac{L}{a_\oplus} \approx \frac{1\,\text{KM}}{6{,}000\,\text{KM}}$$

$\oplus$ ← symbol for Earth

$\epsilon \ll 1$ ⇝ it is quite natural to treat Earth as flat.

Examples from 2D: In the usual Euclidean plane

$$\sum_{\substack{\text{vertices}\\ \text{of a triangle}}} \begin{pmatrix}\text{interior}\\ \text{angle}\end{pmatrix} = \pi$$

$$= \alpha + \beta + \gamma.$$

and

$$\frac{\text{Circum of circle}}{\text{radius}} = \frac{c}{r} = 2\pi$$

Consider instead the surface of a sphere. How do we even define a triangle?

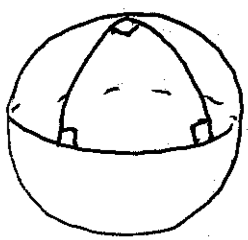
It is a figure consisting of three "straight as possible" sides. Here "straight as possible" is determined by shortest path joining the end points; these are called great circles.

Spherical triangles satisfy

$$\sum_{\text{vertices}} \begin{pmatrix}\text{int.}\\ \text{angle}\end{pmatrix} = \pi + \frac{\text{Area}}{a^2}$$

a = radius of sphere

For example,



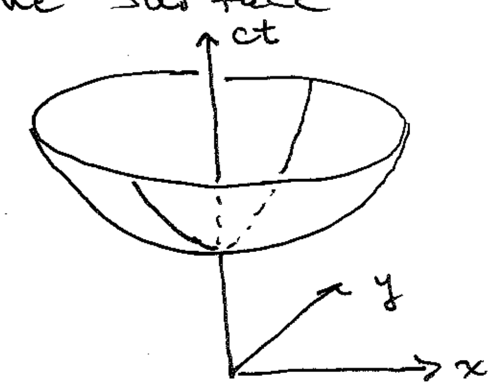
$$\sum_{\text{vertices}} \begin{pmatrix}\text{int.}\\ \text{ang.s}\end{pmatrix} = \frac{3\pi}{2}$$

$$= \pi + \frac{\frac{1}{8}(4\pi a^2)}{a^2} = \frac{3\pi}{2} \checkmark$$

Note: Easiest way to generate a great circle is to intersect a plane through the sphere's center with the sphere.

Another example: Anticipating S.R.

Consider a space w/ coord. (ct, x, y) and the surface



$$-(ct)^2 + x^2 + y^2 = -1, \quad ct > 0$$

or $\quad ct = +\sqrt{1 + x^2 + y^2}$

This is called the upper sheet of the two-sheeted hyperbaloid.

"Straight as possible" lines can again be generated by planes through the origin.

Hyperbolic triangles satisfy

$$\sum_{\text{verts}} \left(\begin{matrix} \text{int.} \\ \text{angs} \end{matrix} \right) = \pi - \frac{\text{Area}}{R^2}$$

↑ Constant radius of curvature of the hyperbaloid.

2nd Revolution (Riemann)

$\boxed{\text{Characterize geometry locally and allow variation from point to point}}$

Main approach: Specify

$$dS \leftarrow \text{inf. dist. btwn near pts.}$$

and use integral and differential calculus to build up finite distances, angles, etc.

- Important insight: This allows one

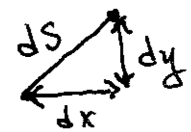
These are just two examples — very special examples in fact, they both have constant curvature — others would include the surface of an egg, peanut, or football.

Already you can probably see how limited our calculational techniques are.

to characterize a geometry intrinsically, that is, without viewing it "from the outside" as a surface in a higher dimensional space.

A surface's extrinsic geometry describes how it sits inside a larger space.

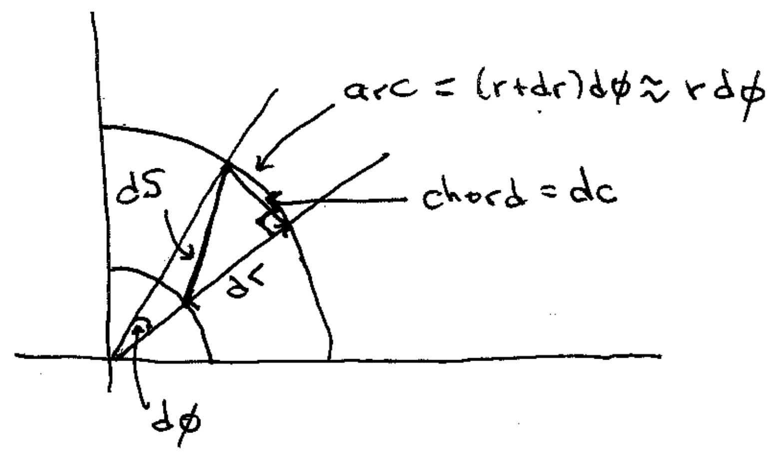
Ex.: Euclidean plane. Coord.s (x,y)



$$dS^2 = dx^2 + dy^2$$

Convention: 'S' for 2D and 3D

Or in coord.s (r, ϕ)



$$dS^2 = dr^2 + dc^2$$

$$dc = 2(r+dr)\sin\left(\frac{d\phi}{2}\right)$$

$$\approx 2r\frac{d\phi}{2} + O(dr\,d\phi, d\phi^2, \dots)$$

So,

$$dS^2 = dr^2 + r^2 d\phi^2$$

An important check is that you get the same formula for $dS(r,\phi)$ by doing a coordinate transformation of $dS(x,y)$; this is good(!), we don't want our physical results to depend on our choice of coord.s.