General Relativity

Day 20 — Mar 16th, 2016

Today

I Last time

II Solving the geodesic Eq. (GE), conservation, & symmetry

III Null geodesics

I • Derived the geodesic equation from

$$L = -g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$$

[Aside: The question arose why is $ds^2 = -d\tau^2$ $(c=1)$?

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

$$= g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}d\tau^2$$

$$= (g_{\alpha\beta}u^\alpha u^\beta)\,d\tau^2$$

$$= -d\tau^2. \checkmark$$

• The GE is

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = 0$$

or

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = 0$$

with the Christoffel symbols,

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\delta\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right)$$

II This is a set of 4 coupled, 2nd order ODEs. Hard to solve in general, one has to turn to numerics. However, conservation laws simplify the solution of these equations.

> **Noether's theorem**
> Conservation laws are in 1-to-1 correspondence with symmetries.

<u>Example</u>: Conservation of momentum follows from translational symmetry:

$$L = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 - V(y)$$ ← No x-dependence which means translations in x preserve L and hence S.

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}) = \frac{d}{dt}(P_x) = 0 \Rightarrow \boxed{P_x = \text{const.}}$$

In G.R. a useful way to capture symmetries is by using a vector, called the generator of the symmetry. is called a <u>Killing vector</u>.

In the flat space example above, we have

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

and $\tilde{x}^\alpha = x^\alpha + a\xi^\alpha \Rightarrow d\tilde{x}^\alpha = dx^\alpha$

$$\Rightarrow d\tilde{s}^2 = -d\tilde{t}^2 + d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2,$$

which is the same metric.

How does a Killing vector lead to a conservation law? Let's use our example to illustrate — a metric invariant under x^1 translations.

For example, the flat space metric is invariant under the translation $x^1 \to x^1 + \text{const}$ and this is captured by the vector

$$\xi^\alpha = (0, 1, 0, 0)$$

that points in the direction of the translation. In G.R. a vector that generates a symmetry, i.e. $\tilde{x}^\alpha = x^\alpha + a\xi^\alpha$ leaves the metric (and action) invariant,

So, L doesn't depend on x^1 and

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial\left(\frac{dx^1}{d\tau}\right)}\right) = 0$$

(recall $L = -g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$)

$$\Rightarrow \frac{d}{d\tau}\left(-g_{\alpha\beta}\,\delta^\alpha_1\,\dot{x}^\beta\right) = 0$$

$$\Rightarrow \frac{d}{d\tau}\left(-g_{1\beta}\dot{x}^\beta\right) = \frac{d}{d\tau}\left(-g_{\alpha\beta}\xi^\alpha u^\beta\right) = 0$$

$$\Rightarrow \boxed{-\xi \cdot u = \text{const.}}$$

when ξ is a Killing vector.

Recall the example, from our last two classes, of geodesics in the plane using polar coords. We found GEs

$$\left.\begin{aligned} \frac{d^2 r}{ds^2} &= r\left(\frac{d\phi}{ds}\right)^2 \\ \frac{d}{ds}\left(r^2 \frac{d\phi}{ds}\right) &= 0 \end{aligned}\right\} \text{tricky to solve.}$$

Instead of attacking these directly we use conservation laws. First, note that in this spatial context the appropriate analog of $\underset{\sim}{u}\cdot\underset{\sim}{u} = -1$ is $\vec{u}\cdot\vec{u} = 1$ and so,

$$\left(\frac{ds}{ds}\right)^2 = 1 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2$$

Also, the metric doesn't depend on ϕ so,

$$l \equiv \vec{\xi}\cdot\vec{u} = g_{AB}\,\xi^A u^B = r^2 \frac{d\phi}{ds}$$

Putting this into the 1st conservation law,

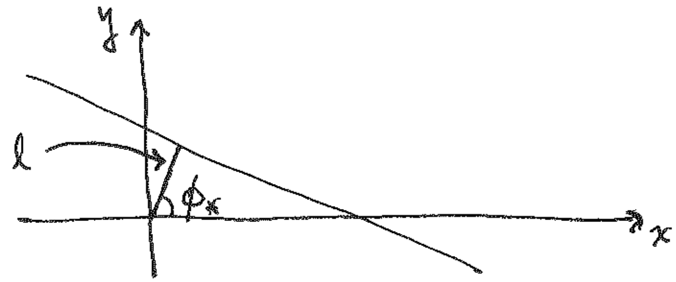
$$\frac{dr}{ds} = \left(1 - \frac{l^2}{r^2}\right)^{1/2}$$

To find the orbit we take,

$$\frac{d\phi/ds}{dr/ds} = \frac{d\phi}{dr} = \frac{l}{r^2}\left(1 - \frac{l^2}{r^2}\right)^{-1/2}$$

$$\Rightarrow \phi = \phi_* + \cos^{-1}\left(\frac{l}{r}\right)$$

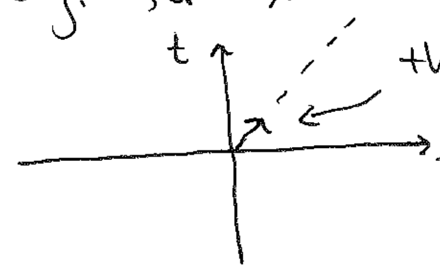
or $$r\cos(\phi - \phi_*) = l$$



III For light $m = 0$, $V = 1$ and

$$ds^2 = -d\tau^2 = 0$$

This means that we can't use s or τ to parametrize light trajectories. Instead, we invent a parametrization — e.g. for $x = t$ we take multiples of this vector. That is,



$$x^\alpha = \lambda\, u_c^\alpha, \quad \text{with}$$

$$u_c^\alpha = (1, 1, 0, 0).$$

... Continued next time.