# General Relativity

## Day 21

Mar 18th, 2016 Pl/3

**Today**

I Last time

II Null geodesics

III Riemann Normal Coordinates

I • Intro'd Noether's thm.:

$\boxed{\text{Conservation laws are in 1-to-1 correspondence w/ symmetries}}$

• In GR symmetries are captured by Killing vectors s.t.

$$\tilde{x}^\alpha = x^\alpha + a\,\xi^\alpha$$

leaves the metric invariant.

• In particular the Noetherian connection comes from

$$\boxed{-\underset{\sim}{\xi}\cdot\underset{\sim}{u} = \text{const.}}$$

• Looked at this in the polar plane:

Analogies: $\tau \to s$, $\underset{\sim}{u}\cdot\underset{\sim}{u} = -1 \to \vec{u}\cdot\vec{u} = 1$

where $\vec{u} = \frac{d\vec{x}}{ds}$. $g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

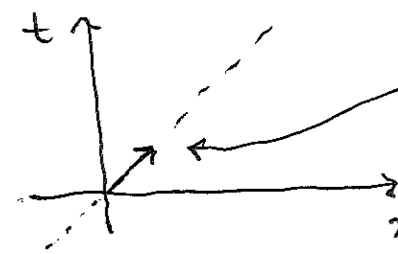
and found

$$\text{Const.} = l = \vec{\xi}\cdot\vec{u} = g_{AB}\xi^A u^B = \boxed{r^2 \frac{d\phi}{ds}}$$

II For light $m=0$, $v = 1 = c$, and

$$ds^2 = -d\tau^2 = 0\,!$$

This means that we can't use $s$ or $\tau$ to parametrize light trajectories. Instead, we invent a parametrization. E.g. for $x = t$ we take multiples of this vector. That is,




$$x^\alpha(\lambda) = \lambda\, u_c^\alpha$$

with $u_c^\alpha = (1, 1, 0, 0)$.

Note that for this parametrization

$$u^\alpha = \frac{dx^\alpha}{d\lambda} = u_c^\alpha$$

and

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1 + 1 + 0 + 0 = 0,$$

i.e. $\underset{\sim}{u}$ is a null vector. Also,

$$\frac{d^2x^\alpha}{d\lambda^2} = \frac{du^\alpha}{d\lambda} = 0$$

just like a free particle. If instead $x^\alpha = \sigma^3 u_c^\alpha$, then the same trajectory would have,

$$u^\alpha = \frac{dx^\alpha}{d\sigma} = (3\sigma^2, 3\sigma^2, 0, 0)$$

and $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$ still, but

$$\frac{d^2x^\alpha}{d\sigma^2} = \frac{du^\alpha}{d\sigma} = 6\sigma u_c^\alpha \neq 0.$$

We call a parametrization of a light ray <u>affine</u> if it gives the free particle eq. for light rays. Affine parameters are preferred, but not unique.

What about the geodesic equation in a general metric?

For light $ds^2 = 0$

$$\Rightarrow \underset{\sim}{u} \cdot \underset{\sim}{u} = g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0,$$

with $\lambda$ an affine parameter.

This is only 1 equation. To derive null geodesics you need relativistic Elec. & Mag., but we can guess,

$$\frac{d^2x^\gamma}{d\lambda^2} + \Gamma^\gamma_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0$$

and it is correct!

<u>Summary</u>: As long as we work with affine parameters for null geodesics, the GE equation always takes the same form

$$\frac{d^2x^\gamma}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0 \quad \text{(timelike)}$$

and

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\gamma_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0 \quad \text{(null.)}$$

III We have been discussing local inertial frames (LIFs), where

$$g_{\alpha\beta}(x_P) = \eta_{\alpha\beta}, \quad \left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_{x=x_P} = 0$$

Note that were we to find such coordinates

$$\left.\Gamma^\alpha_{\beta\gamma}\right|_{x_P} = 0 \Rightarrow \left.\frac{d^2x^\alpha}{d\tau^2}\right|_{x_P} = 0 \Rightarrow \text{straight line motion near } P.$$

This suggests an answer to: How do we construct Riemann Normal coord.s?

- Label points by : $x^\alpha = s n^\alpha$.

That's it! These steps give unique labels until geodesics cross — so we restrict to this neighborhood of P.

Now,

$$g_{\alpha\beta}(x_P) = \eta_{\alpha\beta} \checkmark \quad = \text{an orthonormal basis}$$

we chose our coordinate

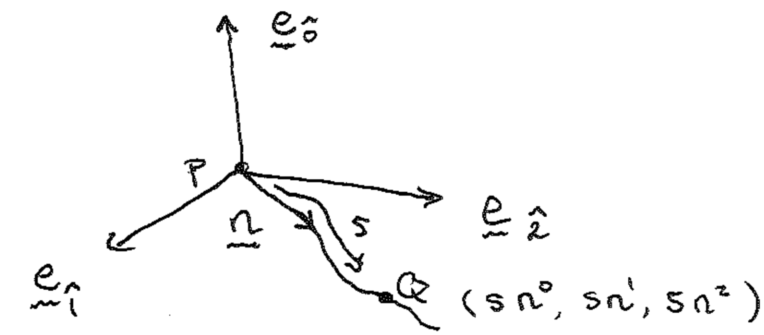
and

$$\frac{d^2x^\alpha}{ds^2} = \frac{d}{ds}\left(\frac{dx^\gamma}{ds}\right) = \frac{d}{ds}(n^\alpha) = 0 = \Gamma^\gamma_{\beta\gamma} n^\beta n^\alpha,$$

but, this holds for all $n^\beta, n^\gamma$, so

Let's do it. The steps:

- Pick P.
- Erect an orthonormal frame at P.

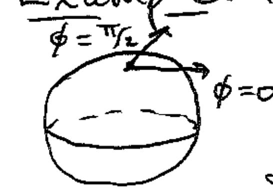



- Follow geodesics a distance s, or proper time $\tau$.

$$\Gamma^\gamma_{\beta\gamma} = 0 \Rightarrow \left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_{x_P} = 0$$

This coord. system, which yields an LIF, is called the Riemann normal coordinate system.

<u>Example</u>: North pole of Sphere again:




$$n^A = (\cos\phi, \sin\phi)$$

and the arc length of a great circle is $s = a\theta$, so,

$$x^A = (a\theta\cos\phi, a\theta\sin\phi)$$

Look familiar?!