General Relativity

Day 22

<u>Today</u>

I Where were we?

II Schwarzschild and Central Forces

III Logistics: problem solving this week and exam: mean: 39.5
STD: 8.

I • We had just completed our treatment of one of the fundamental eqn.s of G.R.:

The geodesic equation:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \quad \text{(timelike)}$$

or

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \quad \text{(lightlike)}$$

These equations describe how particles subjected only to gravity move.

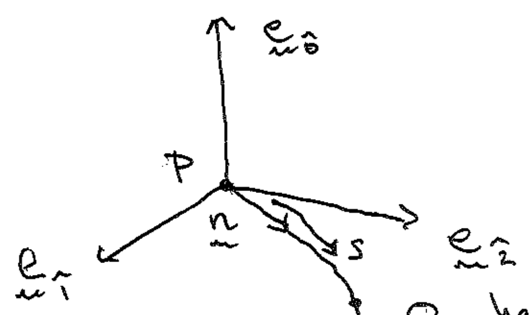
• We took a glimpse at how to solve these equations using conservation laws; these are usually arrived at through symmetry and

$$\boxed{-\underset{\sim}{\xi}\cdot\underset{\sim}{u} = \text{const.}}$$

• Geodesics help us to build a local inertial frame where

$$g_{\alpha\beta}(x_P) = \eta_{\alpha\beta} \quad \text{and} \quad \left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_{x=x_P} = 0$$

Steps: • Pick P.
• Erect an orthonormal frame at P.



Q has coords (sn^0, sn^1, sn^2)

- Follow geodesics a distance S, or a proper time τ.
- Label points by $x^\alpha = S n^\alpha$.

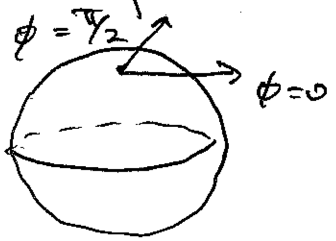
That is all we need. Note that

$$\frac{d^2 x^\alpha}{dS^2} = \frac{d}{dS}\left(\frac{dx^\alpha}{dS}\right) = \frac{d}{dS}(n^\alpha) = 0 = \Gamma^\alpha_{\beta\gamma} n^\beta n^\gamma$$

and this holds for all n^β, n^γ, so

$$\Gamma^\alpha_{\beta\gamma} = 0 \Rightarrow \left.\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right|_{x_p} = 0$$

Example: North pole of sphere again



$n^A = (\cos\phi, \sin\phi)$ and arclength of a great circle is $S = a\theta$.

So, $x^A = (a\theta\cos\phi, a\theta\sin\phi)$.

Look familiar? These are the coordinates that you used to show that the spherical metric becomes η_{AB} to 2nd order in problem class.

It also shows that these are locally "straight lines" in the sense that

$$\frac{d^2 x^\alpha}{dS^2} = 0$$

So,

$$g_{\alpha\beta}(x_p) = \eta_{\alpha\beta} = \text{an orthonormal basis.}$$

These are the Riemann normal coordinates we have been discussing and yields a Local Inertial Frame or LIF.

II. Now we begin our exploration of the Schwarzschild geometry:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

This metric describes the geometry of a spherically symmetric source of curvature. In G.R. mass is a subtle concept; recall our observation that a hot potato weighs more.

Note that for small $\frac{GM}{rc^2}$ we can expand:

$$d\Omega^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2$$

$$ds^2 \approx -\left(1 - \frac{2GM}{rc^2}\right)(c\,dt)^2 + \left(1 + \frac{2GM}{rc^2}\right)dr^2 + r^2 d\Omega^2$$

and this is very similar to

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)(c\,dt)^2 + \left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2)$$

and suggests the identification $\Phi = -\frac{GM}{r}$ and interpretation of M as the mass of the spherically symmetric body. These turn out to be correct at lowest order.

As promised we will explore this geometry by examining its geodesics. Before doing this it will be very useful to remind ourselves of how central forces work in non-relativistic physics, which we will begin with next time.