

General Relativity

Mar 30th, 2016 $\tau^{1/3}$

Today

I Last time

II Schwarzschild: Energy
as Effective Potential

Day 23

I . Reviewed the Riemann Normal coordinates Construction and the example of the Earth (a sphere).

- Introduced the curved geometry of a spherically symmetric mass distribution of total mass M
 - the Schwarzschild geometry.
- The metric for this geometry is:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$
 where

$$d\Omega^2 = (d\theta^2 + \sin^2\theta d\phi^2)$$
 is the metric on the unit sphere.
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To study geodesics we'll use the

associated conserved quantities, which are guaranteed by Noether's theorem, arising from these symmetries:

$$C = -\tilde{e} \cdot \tilde{u} = -g_{\alpha\beta} \tilde{e}^\alpha u^\beta$$

$$= -g_{00} \tilde{e}^0 u^0 = \left(1 - \frac{2\mu}{r}\right) \frac{dt}{d\tau}$$

This is the conserved energy per unit rest mass, m.

It implies that ϕ is always zero and the motion stays in the "plane" $\phi=0$. Let's reorient so that $\theta=\pi/2$ and $u^\theta=0$. This will be what we work with from now on.

Geodesics: As always, we use

$$\tilde{u} \cdot \tilde{u} = -1$$

so,

$$-1 = g_{\alpha\beta} u^\alpha u^\beta = -\left(1 - \frac{2\mu}{r}\right)(u^t)^2 + \left(1 - \frac{2\mu}{r}\right)(u^r)^2 + r^2(u^\theta)^2$$

This is now of the form

$$E = \frac{e^2 - 1}{2} = \frac{1}{2} \left(\frac{dt}{d\tau} \right)^2 + \frac{1}{2} \left[\left(1 - \frac{2\mu}{r}\right) \left(1 + \frac{r^2}{l^2}\right) - 1 \right]$$

Subtract 1 from each side and multiply by l^2 to get

$$E^2 - 1 = \frac{1}{2} \left(\frac{dt}{d\tau} \right)^2 + \frac{1}{2} \left[\left(1 - \frac{2\mu}{r}\right) \left(1 + \frac{r^2}{l^2}\right) - 1 \right]$$

And,

$$\tilde{u} \cdot \tilde{u} = g_{\alpha\beta} \tilde{u}^\alpha \tilde{u}^\beta$$

$$= g_{33} r^3 u^3 = r^2 \sin^2 \frac{d\phi}{d\tau},$$

which is the conserved angular momentum per unit rest mass.

Now, we perform a slightly sneaky coordinate change: if initially $\phi = 0$ and $\frac{d\phi}{d\tau} = 0$ then the conserved

$$\Rightarrow -\left(1 - \frac{2\mu}{r}\right) e^2 + \left(1 - \frac{2\mu}{r}\right) \left(\frac{dt}{d\tau}\right)^2 + \frac{r^2}{l^2} = -1$$

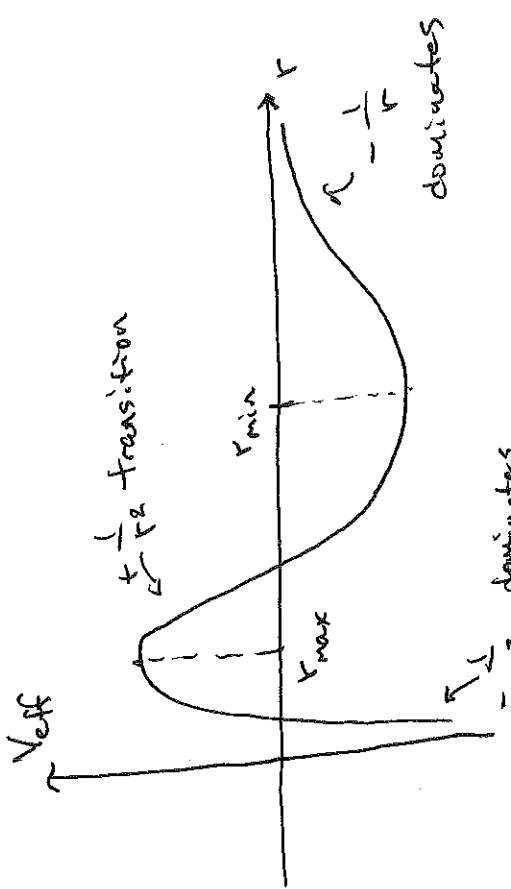
$$\Rightarrow -e^2 + \left(\frac{dt}{d\tau}\right)^2 + \frac{r^2}{l^2} \left(1 - \frac{2\mu}{r}\right) = -\left(1 - \frac{2\mu}{r}\right)$$

$$\Rightarrow e^2 = \left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2\mu}{r}\right) \left(1 + \frac{r^2}{l^2}\right)$$

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$$E^2 - 1 = \frac{1}{2} \left(\frac{dt}{d\tau} \right)^2 + \frac{1}{2} \left[\left(1 - \frac{2\mu}{r}\right) \left(1 + \frac{r^2}{l^2}\right) - 1 \right]$$

Plotting \dot{r} vs r



$$E = \frac{1}{2} \left(\frac{dr}{dt} \right)^2 + V_{\text{eff}}$$

Everything per unit mass) with

$$V_{\text{eff}} = -\frac{N}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}.$$

This is very similar to the effective potential of the Kepler problem that we studied in the problem solving session, except that it includes a relativistic correction $-\frac{ML^2}{r^3}$.