General Relativity — Mar 30^{th}, 2016 — P1/3

Day 23

Today

I Last time

II Schwarzschild: Energy & Effective Potential

I
- Reviewed the Riemann Normal coordinates Construction and the example of the Earth (a sphere).
- Introduced the curved geometry of a spherically symmetric mass distribution of total mass M — the Schwarzschild geometry.

- The metric for this geometry is:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\, d\Omega^2$$

where

$$d\Omega^2 = \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

is the metric on the unit sphere.

II The Schw. geometry has two important symmetries:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

It is static, i.e. independent of time, and hence

$$\xi^\alpha = (1,0,0,0)$$

is a Killing vector. It is also spherically symmetric for constant t and r. One generator is ϕ-translations, i.e.

$$\eta^\alpha = (0,0,0,1)$$

is a Killing vector.

To study geodesics we'll use the associated conserved quantities, which are guarenteed by Noether's theorem, arising from these symmetries:

$$e = -\underset{\sim}{\xi} \cdot \underset{\sim}{u} = -g_{\alpha\beta}\xi^\alpha u^\beta$$

$$= -g_{00}\xi^0 u^0 = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

This is the conserved energy per unit rest mass, m.

l implies that ϕ is always zero and the motion stays in the "plane" $\phi = 0$. Let's reorient so that $\theta = \pi/2$ and $u^\theta = 0$. This will be what we work with from now on.

<u>Geodesics</u>: As always, we use

$$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1$$

So,

$$-1 = g_{\alpha\beta}u^\alpha u^\beta = -\left(1 - \frac{2M}{r}\right)(u^t)^2 + \left(1 - \frac{2M}{r}\right)^{-1}(u^r)^2 + r^2(u^\phi)^2$$

And,

$$l = \underset{\sim}{\eta} \cdot \underset{\sim}{u} = g_{\alpha\beta}\eta^\alpha u^\beta$$

$$= g_{33}\eta^3 u^3 = r^2\sin^2\theta \frac{d\phi}{d\tau},$$

which is the conserved angular momentum per unit rest mass.

Now, we perform a slightly sneaky coordinate change: if initially $\phi = 0$ and $\frac{d\phi}{d\tau} = 0$ then the conserved

$$\Rightarrow -\left(1 - \frac{2M}{r}\right)^{-1}e^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{r^2} = -1$$

$$\Rightarrow -e^2 + \left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right) = -\left(1 - \frac{2M}{r}\right)$$

$$\Rightarrow e^2 = \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right)$$

Subtract 1 from each side and multiply by ½ to get

$$\mathcal{E} \equiv \frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right]$$

This is now of the form

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}$$

(everything per unit mass) with

$$V_{eff} = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}.$$

This is very similar to the effective potential of the Kepler problem that we studied in the problem solving session, except that it includes a new (general) relativistic correction $-\frac{M\ell^2}{r^3}$.

Plotting it we have