General Relativity — Day 24 — Apr 1st, 2016 — p1/3

<u>Today</u>

I Last time

II How can we calculate things? Examples.

III Precession

I • Schwarzschild metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\,d\Omega^2$$

• It has two Killing vectors

$$\xi^\alpha = (1, 0, 0, 0)$$
$$\eta^\alpha = (0, 0, 0, 1)$$

and we found

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}$$ the conserved energy per unit rest mass

and

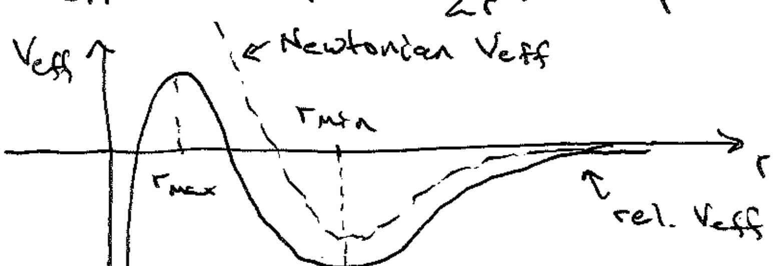
$$\ell = r^2 \sin^2\theta \frac{d\phi}{d\tau}$$ Ang. mom. per unit rest mass

• We derived

$$\mathcal{E} \equiv \frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

with

$$V_{eff} = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$



II <u>Ex1</u> Let's start with r_{min} and r_{max}:

Fix ℓ and calculate

$$\frac{dV_{eff}}{dr} = 0$$

to find

$$r_{\substack{min\\max}} = \frac{\ell^2}{2M}\left[1 \pm \sqrt{1-12\left(\frac{M}{\ell}\right)^2}\,\right]$$

Note that for $\ell < \sqrt{12}\,M$ extrema disappear. ⇒ If you don't have enough angular momentum you fall in.

<u>Ex. 2.</u> Stable circular orbits

Circular orbits can occur at r_{min} and r_{max}, but are only <u>stable</u> at r_{min}.

For given ℓ there's only one r_{min}, but by varying ℓ we can ask for the innermost stable circular orbit (ISCO).

The extrema $r_{\substack{min \\ max}}$ coalesce and disappear when $\ell = \sqrt{12}\, M$ and hence

$$r_{min}(\ell = \sqrt{12M}) \boxed{= r_{ISCO} = \frac{12 M^2}{2M} = 6M}$$

Some algebra gives,

$$\frac{\ell}{e} = (Mr)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1}. \quad \text{for circular orbits}$$

Then,

$$\Omega = \frac{(Mr)^{1/2}}{r^2}$$

or

$$\boxed{\Omega^2 = \frac{M}{r^3}} \quad \text{circular orbits in a Schw. geometry}$$

Kepler's 3rd law holds for circular orbits in G.R.

<u>Ex. 3</u> What's $\Omega = d\phi/dt$ for circular orbits?

Well,

$$\Omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau}$$

$$= \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\left(\frac{\ell}{e}\right).$$

Using

$$r_{min} = \frac{\ell^2}{2M}\left[1 + \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\right]$$

and

$$e^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \ell^2/r^2\right)$$

What is the 4-velocity on such an orbit?

Well, $u^t \Omega = d\phi/d\tau$, so

$$u^\alpha = (u^t, 0, 0, u^t \Omega).$$

As always, $\underset{\sim}{u} \cdot \underset{\sim}{u} = -1 = g_{00} u^{t\,2} + g_{33} u^{t\,2} \Omega^2$

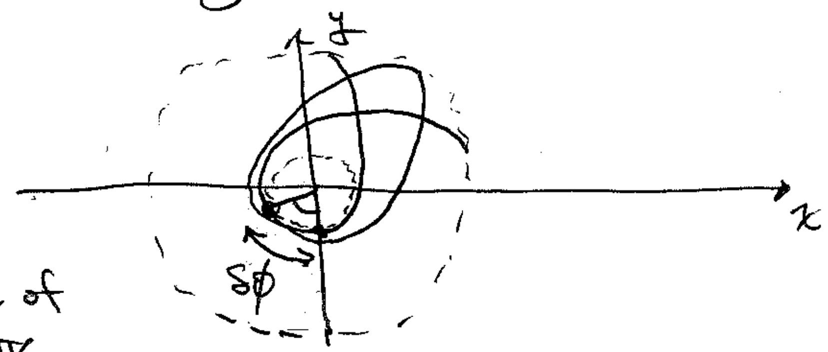
$$= -\left(1 - \frac{2M}{r}\right) u^{t\,2} + r^2 \frac{M}{r^3} u^{t\,2}$$

$$\Rightarrow u^{t\,2} \left(1 - \frac{2M}{r} - \frac{M}{r}\right) = 1$$

or

$$\boxed{u^t = \left(1 - \frac{3M}{r}\right)^{-1/2}} \quad \text{circular orbits.}$$

III A planet like Mercury does not follow a closed elliptical orbit but instead a rich florette like a rotating ellipse



Plane of $\theta = \pi/2$

where does the rest come from?

For closed orbits define

$\Delta\phi$ = angle traversed upon returning to a point of closest approach

$= 2\pi$

For a precessing orbit we are interested in

$$\delta\phi_{prec} = \Delta\phi - 2\pi$$

This is due to a number of effects including the other planets, the oblateness of the sun etc. The measured value is

$$\delta\phi = 574.10 \pm 0.65''/\text{century}$$

Using Newtonian gravity you can account for the other planets and find

$$\delta\phi_{Newton} = 531.65 \pm 0.69''/\text{century}$$

Using the geodesic equation, we can solve for the shape of the orbit

$$\frac{d\phi}{dr} = \frac{\ell}{r^2}\left[e^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)\right]^{-1/2}$$

Separating variables and integrating from the closest approach to the furthest gives

$$\Delta\phi = \underset{\text{to give full orbit}}{2}\,\ell \int_{r_1}^{r_2} \frac{dr}{r^2}\left[c^2(e^2-1) + \frac{2GM}{r} - \frac{\ell^2}{r^2} + \frac{2GM\ell^2}{c^2 r^3}\right]^{-1/2}$$

On the HW you will reproduce Einstein's approximation of this integral and find

$\delta\phi_{GR} = 42.9''/\text{century}$, exactly the discrepancy!