General Relativity

Day 25

April 4th, 2016

<u>Today</u>

I Last time

II Schwarzschild Black Hole

III Light cones

I

- Began the project of exploring Sch. Spacetime by calculating properties of geodesics.
- For example we calculated:
  - r_{min} and r_{max}, the radii of the min and max of V_{eff}
  - The radius of the ISCO r_{ISCO}

and

  - $\Omega = \frac{d\phi}{dt}$ the coordinate angular speed for a circular orbit.

- Set you up to reproduce Einstein's famous calculation of

$$\delta\phi_{precGR} = 42.98 \ \frac{\text{arcsec}}{\text{century}}$$

There are many, many more effects to explore in Schwarzschild, e.g. the bending of star light. We proceed to the study of black holes.

II When a star runs out of fuel there are two possibilities:

- The remaining mass is supported by non-thermal pressures (e.g. Fermi pressures)
- No available pressure can support the remaining mass and the star collapses under its own gravity.

Let's assume spherical collapse.

According to a theorem of Newton's a time dependent mass distribution $\rho_m(t)$, if spherically symmetric, yields a potential that only depends on

$$M = \int \rho_m(t_0)\, d^3x \rightsquigarrow V = -\frac{GMm}{r}.$$

There is a corresponding thm in GR, call Birkhoff's thm (of course, this holds outside the region containing the masses).

Thus, outside the spherical collapse the spacetime only depends on M and is in fact Sch.

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Now, however, we should think more carefully about $r=2M$ and the interior, including $r=0$.

<u>$r = 2M$ is a coord. Singularity</u>

We can demonstrate that a singularity is a coord. sing. by finding <u>any</u> coords in which it does not appear.

The troublesome term in Sch. is the coeff. of dr^2 and we <u>can</u> design coords to get rid of it: these are Eddington - Finkelstein coords

$$(t, r, \theta, \phi) \rightarrow (v, r, \theta, \phi)$$

with

$$t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|$$

Now, we can do an important calculation:

For $r > 2M$:

$$dt = dv - dr - 2M\left(\frac{1}{\frac{r}{2M}-1}\right)\cdot\frac{dr}{2M}$$

$$= dv - \left(1 + \frac{1}{\frac{r}{2M}-1}\right)dr$$

$$= dv - \left(\frac{r/2M}{\frac{r}{2M}-1}\right)dr$$

$$= dv - \left(1-\frac{2M}{r}\right)^{-1} dr$$

Then,

$$dt^2 = dv^2 + \left(1-\frac{2M}{r}\right)^{-2} dr^2 - 2\left(1-\frac{2M}{r}\right)^{-1} dv\,dr$$

and the line element becomes

$$ds^2 = -\left(1-\frac{2M}{r}\right)dv^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + 2\,dv\,dr + r^2 d\Omega^2$$

$$= -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

No singularity at $r = 2M$! This shows that $r = 2M$ was always just a coord. singularity. In contrast, we will eventually show that the curvature goes to ∞ at $r = 0$ and that no coord.s can remove this singularity.

III To simplify things let's focus on radial light rays for which $d\theta = d\phi = 0$. These have

$$ds^2 = 0 = -\left(1-\frac{2M}{r}\right)dv^2 + 2\,dv\,dr$$

There are three types of solution:

(i) $dv = 0 \Rightarrow v = \text{const.}$

But, to keep v const. while t increases, r must decrease $\Rightarrow$ these are radially infalling light rays

(ii) If $dv \neq 0 \Rightarrow -\left(1-\frac{2M}{r}\right)dv + 2\,dr = 0$

or

$$dv = \frac{2\,dr}{\left(1-\frac{2M}{r}\right)}$$

But, we just took the derivative of something that gave us $\left(1-\frac{2M}{r}\right)^{-1}$, so

$$v = 2\left(r + 2M\log\left|\frac{r}{2M}-1\right|\right) + \text{const.}$$

We'll return to these in a moment.

(iii) If $r = 2M$ then $dr = 0$ and the coeff. of dv is zero. These are stationary light rays, stuck at $r = 2M$.

Returning to the solns (ii), how do these behave? Are they ingoing or outgoing? Well, for these soln.s

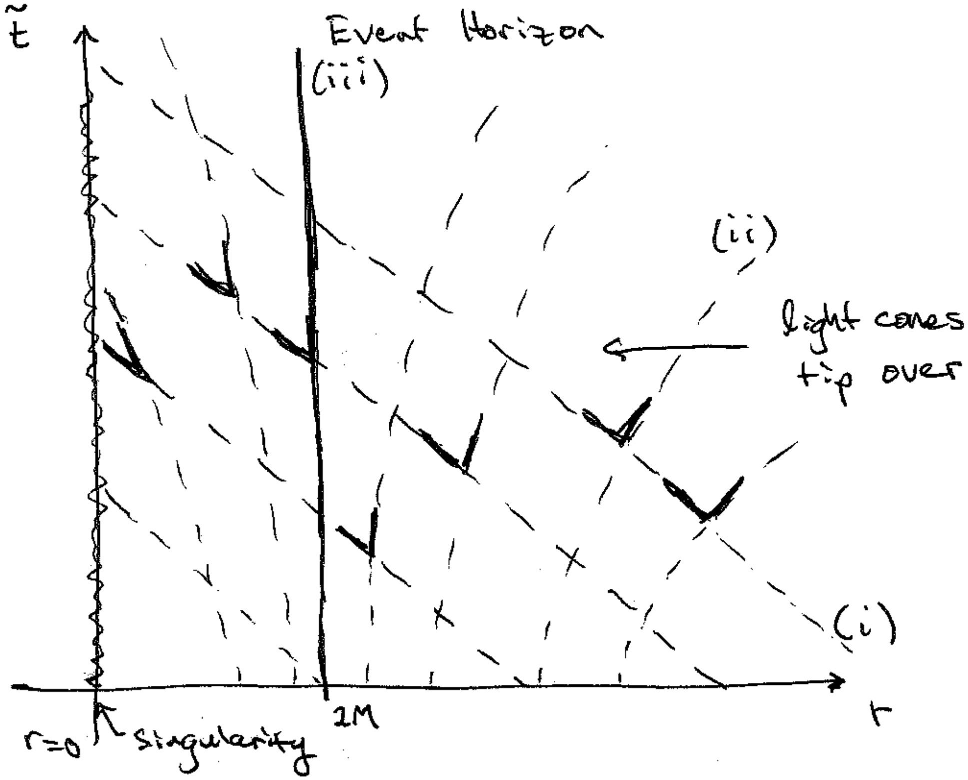
$$t = r + 2M\log\left|\frac{r}{2M}-1\right| + \text{const.}$$

and for increasing t we have two cases:

outgoing when $r > 2M$

ingoing when $r < 2M$

Let's plot 'em: let $\tilde{t} = v - r$ then rays of type (i) are 45° lines with $\tilde{t} = -r + \text{const.}$



For large r, the log is small and

$$t = r + \text{const}.$$

$\Rightarrow$ at large r, the outgoing ray is at $+45^\circ$, like in flat spacetime.