General Relativity

Day 28

Today

I Vectors: a new basis

II Dual vectors

Last time we discussed (i), I relatively informally said: the key is to separate directions & magnitudes. Directions are accesible locally and then we *impose* linearity i.e.,

$$\alpha(\underset{\sim}{a}+\underset{\sim}{b}) = \alpha\underset{\sim}{a} + \alpha\underset{\sim}{b}$$

to build up larger magnitude vectors. We turn now to a careful definition of *directions* and what is meant by *locally*.

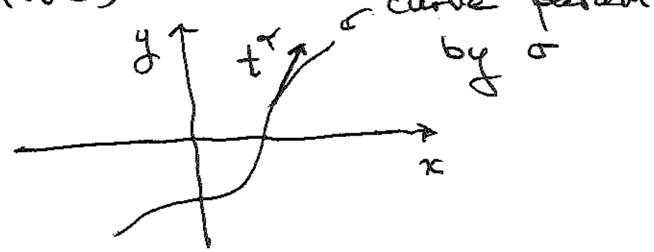
I. Recall our previous discussion of vectors:

(i) How does a local observer talk about (i.e. measure) vectors?

(ii) Where *do* vectors live?

(iii) Does the answer to (ii) mean that vectors based at different pts of our space live in different tangent spaces?

The definition of directions is, at first unexpected and later gratifying; we use directional derivatives:



Consider $f(x^\alpha(\sigma))$ then the derivative of f in the direction of the curve is:

$$\frac{df}{d\sigma} = \lim_{\epsilon \to 0} \left[\frac{f(x^\alpha(\sigma+\epsilon)) - f(x^\alpha(\sigma))}{\epsilon} \right]$$

$$= \frac{\partial f}{\partial x^\alpha} \frac{dx^\alpha}{d\sigma}$$

The part

$$\frac{dx^\alpha}{d\sigma} \equiv t^\alpha$$

is a <u>vector</u> tangent to the curve at σ. Specifying the tangent vector specifies the directional derivative, $\frac{d}{d\sigma} = t^\alpha \frac{\partial}{\partial x^\alpha} = \frac{dx^\alpha}{d\sigma}\frac{\partial}{\partial x^\alpha}$,

and visa versa, as we've seen in the definition of $df/d\sigma$ above. This also explains what I meant by locally — the whole construction takes place in the limit $\epsilon \to 0$.

Q: What derivative is associated with $\vec{x} = \underset{\sim}{e}_1$?

The derivative $\partial/\partial x = \partial/\partial x^1$ is a natural guess because it measures changes in the x^1-direction and indeed

$$\partial/\partial x = \frac{\partial}{\partial x^1} = (0,1,0,0)^\alpha \frac{\partial}{\partial x^\alpha}$$

$$= (\underset{\sim}{e}_1)^\alpha \frac{\partial}{\partial x^\alpha}$$

So the guess is correct!

Q: More generally, what derivative is associated to $\underset{\sim}{e}_\alpha$? Same idea, it's $\partial/\partial x^\alpha$, which we also write as,

$$\underset{\sim}{e}_\alpha = (\underset{\sim}{e}_\alpha)^\beta \frac{\partial}{\partial x^\beta} = \delta^\beta_\alpha \frac{\partial}{\partial x^\beta} = \frac{\partial}{\partial x^\alpha}\ !$$

Q: what does a general vector $\underset{\sim}{a}$ become as a derivative?

From the last question

$$\underset{\sim}{a} = a^\alpha \underset{\sim}{e}_\alpha = a^\alpha \frac{\partial}{\partial x^\alpha}.$$

The <u>tangent space</u> is the linear space of directional derivatives.

<u>Transformations</u>: A recurring question for us has been "how do objects change when we change coordinates?"

Transformation of vectors:

$$\underset{\sim}{a} = a^{\alpha} \frac{\partial}{\partial x^{\alpha}} = a^{\alpha} \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \frac{\partial}{\partial x'^{\beta}}$$

$$\equiv a'^{\beta} \frac{\partial}{\partial x'^{\beta}}$$

So the components in the new coordinates are,

$$\boxed{a'^{\beta} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} a^{\alpha}}$$

Similarly

$$a'^{\alpha} \frac{\partial}{\partial x'^{\alpha}} = a'^{\alpha} \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}} = a^{\beta} \frac{\partial}{\partial x^{\beta}}$$

So,

$$\boxed{a^{\beta} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} a^{\alpha}.}$$

Dual Vectors

"Sophisticated" definition: "Dual vectors eat vectors and return numbers" or "A dual vector $\underset{\sim}{\omega}$ is a linear map from vectors to real numbers."

Linearity, i.e. $\omega(\alpha \underset{\sim}{a} + \beta \underset{\sim}{b}) = \alpha\,\omega(\underset{\sim}{a}) + \beta\,\omega(\underset{\sim}{b})$,

is a tight restriction, most general such map is

$$\omega(\underset{\sim}{a}) = \omega_{\alpha} a^{\alpha}$$

with numbers ω_{α}, called components.

Physics example: Momentum eats velocity and returns twice the kinetic energy:

from its def.:

$$P_i = \frac{\partial L}{\partial \dot{q}^i}$$ lower index.

$$P(\underset{\sim}{v}) = P_i v^i = m v_i v^i = 2\,\text{K.E.}$$

Notice that momentum is naturally dual

Math example: The Gradient of f_x is a dual vector because,

$$\frac{\partial f}{\partial x^{\alpha}} t^{\alpha} = \text{number},$$

it takes $\underset{\sim}{t}$ to a number. We often write it as

$$\underset{\sim}{\nabla} f$$

and it is a dual vector.