General Relativity

Problem Solving 8

April 12th, 2016

Today

I Dual vectors

II The metric revealed

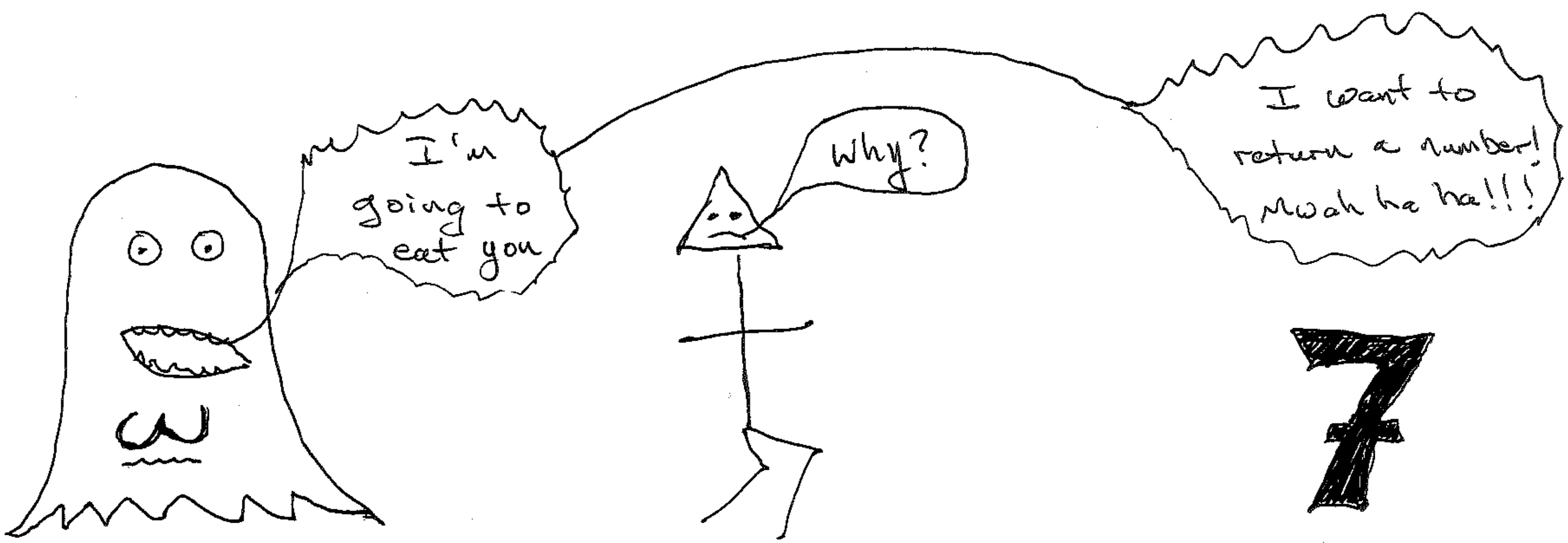
III Problem solving Sheet 8

I A dual vector is a linear map that takes a vector as input and returns a number.

Associated to the dual vector $\underset{\sim}{\omega}$ is the linear map

$$\omega(\underset{\sim}{a}) = \omega_\alpha a^\alpha$$

We call ω_α the components of $\underset{\sim}{\omega}$.



and so we can also introduce a basis for dual vectors, call it $\{\underset{\sim}{e}^{\alpha}\}$ $\alpha = 0, 1, 2, 3$. This allows us to decompose,

$$\underset{\sim}{\omega} = \omega_{\alpha}\, \underset{\sim}{e}^{\alpha}.$$

The dual basis vectors $\underset{\sim}{e}^{\alpha}$ are also maps, they're defined by,

$$e^{\alpha}(\underset{\sim}{e}_{\beta}) \equiv \delta^{\alpha}_{\beta} \equiv \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

Consistency check:

$$\omega(\underset{\sim}{a}) = \omega_{\alpha}\, e^{\alpha}(a^{\beta} \underset{\sim}{e}_{\beta})$$

$$= \omega_{\alpha}\, a^{\beta}\, e^{\alpha}(\underset{\sim}{e}_{\beta})$$

$$= \omega_{\alpha}\, a^{\beta}\, \delta^{\alpha}_{\beta} = \omega_{\alpha} a^{\alpha}. \checkmark$$

But this means $\underset{\sim}{a}$ can be thought of as a dual vector! How?

$$a(\underset{\sim}{b}) = a_{\alpha} b^{\alpha} = \underset{\sim}{a} \cdot \underset{\sim}{b} = g_{\beta\alpha} a^{\beta} b^{\alpha}$$

$$= (g_{\alpha\beta} a^{\beta})\, b^{\alpha}$$

So,

$$\boxed{a_{\alpha} \equiv g_{\alpha\beta}\, a^{\beta}}$$

Any vector (e.g. $\underset{\sim}{a}$) can be converted into a dual vector by using the metric to <u>lower</u> an index.

<u>The secret of the metric revealed:</u>

We already have a linear map of vectors defined by $\underset{\sim}{a}$

$$a(\underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{b}$$

Recall our definition of the inverse metric

$$\boxed{g^{\alpha\gamma} g_{\gamma\beta} = \delta^{\alpha}_{\beta}.}$$

We can use the inverse metric to raise indices, i.e. convert dual vectors to vectors:

$$\boxed{a^{\alpha} = g^{\alpha\beta} a_{b}.}$$

Because of these conversion processes, physicists often identify vector and dual vectors and just speak about the contra- (upper) and co-variant (lower) components of a "vector" (Mnemonic: Co is down low).