General Relativity

Day 29

April 13th, 2016

<u>Today</u>

I Last time

II Tensors

(i) Intro & definition

(ii) Tensors are ubiquitous

(iii) Map notation

I • The <u>tangent space</u> is the linear space of directional derivatives

• It has a coordinate basis

$$\underset{\sim}{e}_\alpha = \frac{\partial}{\partial x^\alpha}.$$

• A dual vector is a linear map from vectors to real numbers.

• Just like ~~vectors~~, dual vectors have components. These are just the numbers that result when a dual vector eats a basis vector,

$$\omega(\underset{\sim}{e}_\alpha) = \omega_\beta e^\beta(\underset{\sim}{e}_\alpha) = \omega_\beta \delta^\beta_\alpha = \omega_\alpha.$$

our definition of dual bases

• The metric allows us to identify dual vectors and vectors: "raising and lowering indices"

$$\boxed{a_\alpha = g_{\alpha\beta} a^\beta \quad , \quad a^\alpha = g^{\alpha\beta} a_\beta}$$

II Tensors are unnecessarily shrouded in a fog of confusion: "... baffling beasts bristling with indices...". The idea of a tensor is actually quite close to what we've just described for dual vectors. A tensor is something that eats vectors (and dual vectors) and returns a number.

They do this in a <u>multilinear</u> manner, i.e. linear in each entry. In equations,

$$t(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) = t(a^\alpha \underset{\sim}{e}_\alpha, b^\beta \underset{\sim}{e}_\beta, c^\gamma \underset{\sim}{e}_\gamma)$$

$$= a^\alpha b^\beta c^\gamma \, t(\underset{\sim}{e}_\alpha, \underset{\sim}{e}_\beta, \underset{\sim}{e}_\gamma)$$

$$= a^\alpha b^\beta c^\gamma \, t_{\alpha\beta\gamma} \leftarrow \text{the components of the tensor}$$

We call the number of vectors and dual vectors that a tensor eats its rank (= total # of indices on tensor).

<u>Tensors are ubiquitous</u>

Why?! I can offer three answers:

(1) Linearity is an effective initial physical model.

Consider a spring: $F = -kx$ is an excellent approximation for small displacements in one dimension;

Things get richer when you consider a 3D balloon. Name three orthogonal faces, say the x-face, y-face and z-face.

We've been spending a lot of time with a particular rank 2 tensor:

$$g(\underset{\sim}{a}, \underset{\sim}{b}) = g(a^\alpha \underset{\sim}{e}_\alpha, b^\beta \underset{\sim}{e}_\beta)$$

$$= a^\alpha b^\beta g(\underset{\sim}{e}_\alpha, \underset{\sim}{e}_\beta)$$

$$= a^\alpha b^\beta g_{\alpha\beta}$$

$$= \underset{\sim}{a} \cdot \underset{\sim}{b}$$

the metric! Claim:

If I push on the x-face a small amount, it squashes a little, and the force the balloon exerts back is, to a good first approx., linear in this displacement.

However, if I sheer the x-face too this ~~shear~~ can in general change the outward force on the x-face.

So, in general

$$F_x = K_{xx} X + K_{xy} Y + K_{xz} Z$$

For all three faces

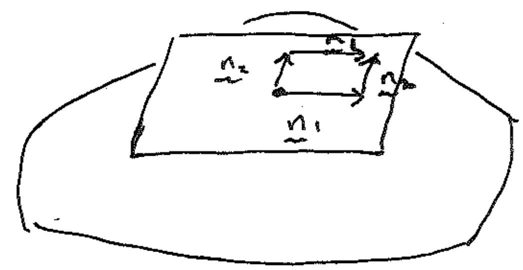
$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \begin{pmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or

$$F_i = k_{ij} x^j$$

The coefficients k_{ij} are called the elasticity tensor. So, again, any time linearity is a good model tensors will show up.

(2) Linearity is computationally nice. We can solve linear systems. We can do calculus with tensors:



Let δ, ϵ be small parameters, then

$$R(\underset{\sim}{a}, \delta \underset{\sim}{n_1}, \epsilon \underset{\sim}{n_2}) = \epsilon \delta R(\underset{\sim}{a}, \underset{\sim}{n_1}, \underset{\sim}{n_2}),$$

and it is easy to consider the limits $\delta, \epsilon \to 0$.

(3) Tensors are great for capturing a large class of invariants. More on this shortly.

Map Notation

Call the vector space that we've been drawing vectors from V (e.g. the tangent space at P.). Call its dual space V^*. Then in map notation

$$t : V \times V \times V \to \mathbb{R}$$

$$(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}) \mapsto a^\alpha b^\beta c^\gamma t_{\alpha\beta\gamma}$$

This notation helps us to address one of your long standing questions: Is the metric a matrix or not?

Recall, a linear map is a map

$$M : V \to V$$

such that M is linear

TO BE CONTINUED . . .