General Relativity   Feb 5th, 2016   P1/4

Today

I Last time

II Geodesics

III The Chain Rule

IV Calculus of variations

I

- Because

  inertial mass = gravitational mass

  gravity is about geometry.

- Two revolutions

  1st: There exists more than one geometry.

  2nd: Characterize geometry locally and allow variation from point to point

- Main tool: The line element, e.g.

$$dS^2 = dr^2 + r^2 d\phi^2$$

II Through investigating spherical triangles we have already seen how interesting geodesics can be. Several of you have asked how we can calculate which curves are geodesics, a very natural question.

The answer lies with dS.

Consider a curve

We can calculate its length by breaking it into small straight pieces and adding 'em up. This is precisely

$$\underset{\text{length of curve}}{S} = \int_A^{B \leftarrow \text{fixed end points}} \underset{\text{length of segments}}{dS}$$

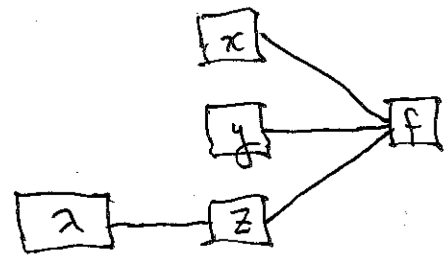
Geodesics (1st definition): Curves of extremal length in a given geometry.

Interesting! This means that to calculate them we want to find the max, min, or saddle of an integral. This (mathematical) problem is what we turn to now. First some background on the chain rule.

III Consider $f = f(x, y, z)$ and the case in which $z = z(\lambda)$, so that

$$f = f(x, y, z(\lambda))$$

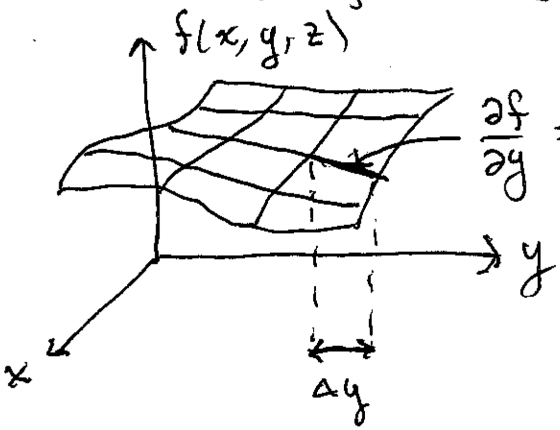
Schematically



If we shake (that is, vary) y then

$$\frac{\partial f}{\partial y}$$

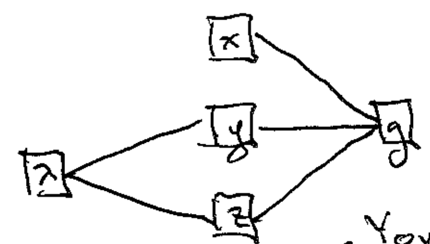
gives how f changes when you vary y keeping all other variables fixed, graphically



$\frac{\partial f}{\partial y}$ = slope of tangent line

What about if you shake λ? Causes z to shake and $\frac{df}{d\lambda} = \frac{\partial f}{\partial z}\frac{dz}{d\lambda}$

Two subtleties: (1) What if $g = g(x, y(\lambda), z(\lambda))$?



You add the contributions,

$$\frac{d}{d\lambda}(g) = \frac{\partial g}{\partial y}\frac{dy}{d\lambda} + \frac{\partial g}{\partial z}\frac{dz}{d\lambda}$$

If $\frac{dy}{d\lambda} = 0$ you recover previous example.

(2) What if $f = f(x, y, z(\lambda))$ <u>and</u> we know $z(\lambda) = \lambda^2$?

You might like to write

$$f = f(x, y, z(\lambda)) = f(x, y, \lambda^2).$$

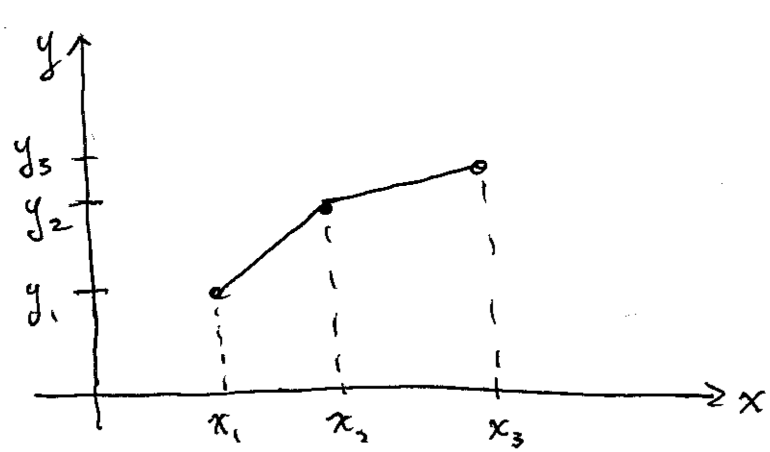
It is unclear to write

$$\frac{d(f)}{d\lambda} = \frac{\partial f}{\partial \lambda^2}\frac{d\lambda^2}{d\lambda} \quad \text{(Bad idea!)}$$

Instead, it is better to write

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial z}\frac{dz}{d\lambda} \overset{\text{New info}}{=} \frac{\partial f}{\partial z}\cdot(2\lambda).$$

them depends on the intermediate pts – let's choose just one (x_2, y_2).



Sometimes we write

$$\frac{\partial f}{\partial z} \equiv D_3 f(x, y, z)$$

to avoid any confusion.

IV First an example: What is the shortest path connecting two fixed points in the x-y plane?

Fix (x_1, y_1) and (x_3, y_3), the length of the path connecting

This length is

$$L = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} + \sqrt{(x_3-x_2)^2+(y_3-y_2)^2}$$

$$\equiv \sqrt{\Delta x^2+(y_2-y_1)^2} + \sqrt{\Delta x^2+(y_3-y_2)^2}$$

We want to choose y_2 such that L is a minimum. So, we impose

$$\frac{\partial L}{\partial y_2} = 0$$

$$\Rightarrow \frac{1}{2}\frac{1}{\sqrt{\Delta x^2+(y_2-y_1)^2}}\cdot 2(y_2-y_1) + \frac{1}{2}\frac{1\cdot 2(y_3-y_2)\cdot(-1)}{\sqrt{\Delta x^2+(y_3-y_2)^2}} = 0$$

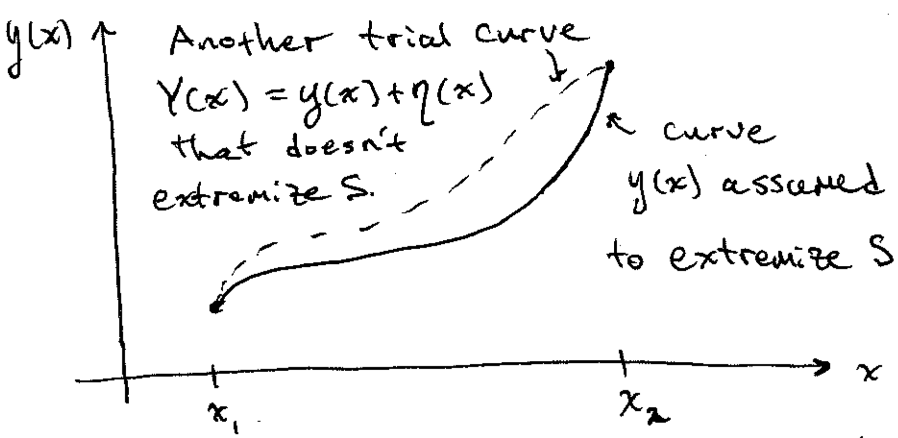
$$\Rightarrow \frac{(y_2-y_1)}{\sqrt{\Delta x^2+(y_2-y_1)^2}} = \frac{(y_3-y_2)}{\sqrt{\Delta x^2+(y_3-y_2)^2}}$$

Factor out Δx^2 to find

$$\frac{(y_2-y_1)/\Delta x}{\sqrt{1-(y_2-y_1)^2/\Delta x^2}} = \frac{(y_3-y_2)/\Delta x}{\sqrt{1-(y_3-y_2)^2/\Delta x^2}}$$

and we get equality if

$$\left(\frac{y_2-y_1}{\Delta x}\right) = \left(\frac{y_3-y_2}{\Delta x}\right)$$



Lagrange also wants to turn this into a calculus problem. He found a clever way to do it:

Write your trial path as

that is, if the slope is const.! It's a straight line!!

We'd like to do this allowing arbitrary changes of the intermediate points.

Lagrange's Method

We want to find the curve $y(x)$ that extremizes

$$S = \int_{x_1}^{x_2} L(x, y(x), y'(x))\, dx$$

$$Y(x) = y(x) + \alpha\,\eta(x) \equiv y(x) + \delta y(x)$$

in terms of a parameter α. Now, you can turn on a variation of the whole curve just by changing α. In particular, S becomes a normal function of α, $\mathcal{S}(\alpha)$.

So, the condition that S is extremized is just

$$\frac{dS}{d\alpha} = 0$$